Reconciling OWL and Non-monotonic Rules for the Semantic Web

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Abstract. We propose a description logic extending $\mathcal{SROIQ}$ (the description logic underlying OWL 2 DL) and at the same time encompassing some of the most prominent monotonic and non-monotonic rule languages, in particular Datalog extended with the answer set semantics. Our proposal could be considered a substantial contribution towards fulfilling the quest for a unifying logic for the Semantic Web. As a case in point, two non-monotonic extensions of description logics considered to be of distinct expressiveness until now are covered in our proposal. In contrast to earlier such proposals, our language has the "look and feel" of a description logic and avoids hybrid or first-order syntaxes.

1 Introduction

The landscape of ontology languages for the Semantic Web is diverse and controversial [11]. In terms of expressive ontology representation formalisms, this is most clearly reflected by the two W3C standards RIF [15] and OWL [10], where the former—the Rule Interchange Format—is based on rules in the wider sense including Datalog and logic programming [12], and the latter—the Web Ontology Language—is based on description logics [1]. In terms of both academic research and industrial development, these two formalisms cater to almost disjoint subcommunities.

While the different paradigms often focus on different perspectives and needs, the field and its applications would as a whole benefit if a certain coherence were retained. This coherence could be achieved through the use of a *unifying logic*, one which reconciles the diverging paradigms of the Semantic Web stack.[4] Indeed, several proposals have been made towards creating such a unified logic, but the quest remains largely unfulfilled.

Two major rifts have been identified which need to be overcome for a reconciliation to occur. This is particularly true in the case of OWL and rule languages. The first rift is due to a fundamental conceptual difference in how the paradigms deal with *unknown* information. While OWL adheres to the so-called *Open World Assumption* (OWA), and thus treats unknown information indeed as unknown, rules and logic programming adhere to the so-called *Closed World Assumption* (CWA) in which unknown information defaults to *false*, i.e., the knowledge available is thought to be a complete encoding of the domain of interest. The second rift is caused by the decision to design OWL as a decidable language: Naive combinations (e.g., in first-order predicate logic) of rule languages (such as Datalog) and OWL are undecidable, and thus violate this design decision.

In principle, the second rift cannot be completely overcome. Nevertheless, several decidable combinations of OWL and rules have been proposed in the literature, usually resulting in a *hybrid* formalism mixing the syntax and sometimes even the semantics of rules and description logics (see the survey sections of [18, 17]). The most recent proposal [20] rests on the introduction of a new syntax construct to description logics, called *nominal schemas*, which results in a description logic which seamlessly—syntactically and semantically—incorporates binary DL-safe Datalog [24] (and as we will see in Section 4.1, even incorporates unrestricted DL-safe Datalog).

While we would argue that the introduction of nominal schemas constitutes a major advance towards a reconciliation of OWL and rules, expanding OWL with nominal schemas by itself does nothing to resolve the first of the rifts mentioned above. And so the approach described in [20] can only be viewed as a partial reconciliation.

The main purpose of the present paper is to build upon the work of [20], addressing that first rift. We show that nominal schemas allow not only for a concise reconciliation of OWL and Datalog, but also that the integration can in fact be lifted to cover established closed world formalisms on both the OWL and the rules side. More precisely, we endow $\mathcal{SROIQ}$, the description logic underlying OWL 2 DL, with both nominal schemas and a generalized semantics based on the logic of minimal knowledge and negation as failure (MKNF) [7, 22]. The latter, and thus also the extension of $\mathcal{SROIQ}$ based on it, is non-monotonic and captures both open and closed world modeling. We show that it in fact encompasses major ontology modeling paradigms, including (trivially) OWL 2 DL and its tractable fragments [10] but also unrestricted DL-safe Datalog, MKNF-extended $\mathcal{ALC}$ [7], hybrid MKNF [23], description logics with defaults [2], and the answer set semantics for Datalog with negation [9]. This means it also covers various ways of expressing the closure of concepts and roles, and also of expressing integrity constraints.

The plan of the paper is as follows. In Section 2 we introduce the syntax and semantics of our new logic, called $\mathcal{SROIQV}(\mathcal{B}^s,\times)\mathcal{K}_{\mathcal{NF}}$, and we show that decidable reasoning in a sufficiently large fragment of it can be realized. Section 3 contains examples that illustrate modeling in $\mathcal{SROIQV}(\mathcal{B}^s,\times)\mathcal{K}_{\mathcal{NF}}$. In Section 4, we show how $\mathcal{SROIQV}(\mathcal{B}^s,\times)\mathcal{K}_{\mathcal{NF}}$ encompasses many of the well-known languages and proposals related to the integration of OWL, rules, and non-monotonicity. Section 5 concludes with brief remarks on related work and a discussion of future work.

2 MKNF DL $\mathcal{SROIQV}(\mathcal{B}^s,\times)\mathcal{K}_{\mathcal{NF}}$

Our work is based on the description logic (*DL*) $\mathcal{SROIQV}(\mathcal{B}^s,\times)$. It extends $\mathcal{SROIQ}$ [11, 13] with concept products [19] and Boolean constructors over simple roles [25]. Importantly, it also incorporates nominal schemas [20]. These represent variable nominals that can

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only bind to known individuals. It has been shown that none of these extensions affect the worst case complexity of reasoning in $\mathcal{SROIQ}$ [20, 19]. We refer to [1, 11] for a detailed account on DLs in general and to [20] for $\mathcal{SROIQV}(\mathcal{B}^s,\times)$ in particular.

Following the work in [7], where the DL $\mathcal{ALC}$ is augmented with two *modal operators* $\mathbf{K}$ and $\mathbf{A}$, we define such an extension for $\mathcal{SROIQV}(\mathcal{B}^s,\times)$. The modal operator $\mathbf{K}$ is interpreted in terms of minimal knowledge, while $\mathbf{A}$ is interpreted as autoepistemic assumption and corresponds to $\neg\mathbf{not}$, i.e., the classical negation of the negation as failure operator $\mathbf{not}$ used in [22] instead of $\mathbf{A}$. The extension to $\mathcal{SROIQV}(\mathcal{B}^s,\times)$ is non-trivial in so far as this DL is significantly more expressive than $\mathcal{ALC}$ and, in particular, the usage of modal operators in role expressions is considerably more advanced.

We introduce the syntax and semantics of our proposed language, called $\mathcal{SROIQV}(\mathcal{B}^s,\times)\mathcal{K}_{\mathcal{NF}}$ in the nomenclature introduced in [7], and we provide some remarks on a reasoning procedure.

2.1 Syntax

We consider a signature $\Sigma = \langle N_I, N_C, N_R, N_V\rangle$ where N_I, N_C, N_R, and N_V are pairwise disjoint and finite sets of *individual names*, *concept names*, *role names*, and *variables*. Role names are divided into disjoint sets of simple role names N_R^s and non-simple role names N_R^n. In the following, we assume that Σ has been fixed. We define concepts and roles in $\mathcal{SROIQV}(\mathcal{B}^s,\times)\mathcal{K}_{\mathcal{NF}}$ as follows.

Definition 1 *The set of $\mathcal{SROIQV}(\mathcal{B}^s,\times)\mathcal{K}_{\mathcal{NF}}$ concepts* C *and (*simple/non-simple*) $\mathcal{SROIQV}(\mathcal{B}^s,\times)\mathcal{K}_{\mathcal{NF}}$ roles* R *(*R^s/R^n*) are defined by the following grammar.*

$$\begin{aligned}
\mathsf{R}^s &::= N_R^s \mid (N_R^s)^- \mid U \mid N_C \times N_C \mid \neg\mathsf{R}^s \mid \mathsf{R}^s \sqcap \mathsf{R}^s \mid \mathsf{R}^s \sqcup \mathsf{R}^s \mid \\
&\quad\ \mathbf{K}\mathsf{R}^s \mid \mathbf{A}\mathsf{R}^s\\
\mathsf{R}^n &::= N_R^n \mid (N_R^n)^- \mid U \mid N_C \times N_C \mid \mathbf{K}\mathsf{R}^n \mid \mathbf{A}\mathsf{R}^n\\
\mathsf{R} &::= \mathsf{R}^s \mid \mathsf{R}^n\\
\mathsf{C} &::= \top \mid \bot \mid N_C \mid \{N_I\} \mid \{N_V\} \mid \neg\mathsf{C} \mid \mathsf{C} \sqcap \mathsf{C} \mid \mathsf{C} \sqcup \mathsf{C} \mid\\
&\quad\ \exists\mathsf{R}.\mathsf{C} \mid \forall\mathsf{R}.\mathsf{C} \mid \exists\mathsf{R}^s.\mathsf{Self} \mid \leqslant k\,\mathsf{R}^s.\mathsf{C} \mid \geqslant k\,\mathsf{R}^s.\mathsf{C} \mid \mathbf{K}\mathsf{C} \mid \mathbf{A}\mathsf{C}
\end{aligned}$$

In the above, U is the *universal role*, $\top$ and $\bot$ are the top and bottom concepts, and k is a non-negative integer. Concepts of the form $\{a\}$ with $a \in N_I$ are called *nominals*, while concepts of the form $\{x\}$ with $x \in N_V$ are called *nominal schemas*. The set of all *concept products* $\mathsf{R}_{C\times D}$, *inverse roles* R^-, and the related function $\mathrm{I}nv : \mathsf{R} \to \mathsf{R}$ are all defined as in [20] except that $C, D \in N_C$.

Definition 2 *Given roles $R, S_i \in \mathsf{R}$, a generalized* role inclusion axiom (RIA) *is a statement of the form $S_1 \circ \cdots \circ S_k \sqsubseteq R$, where $R \in \mathsf{R}^s$ only if $k = 1$ and $S_1 \in \mathsf{R}^s$. A set of RIAs is* regular *if there is a strict partial order $\prec$ on R such that*

- *if $R \notin \{S, Inv(S)\}$, then $S \prec R$ if and only if $Inv(S) \prec R$; and*
- *every RIA has the form $R \circ R \sqsubseteq R$, $Inv(R) \sqsubseteq R$, $R \circ S_1 \circ \cdots \circ S_k \sqsubseteq R$, $S_1 \circ \cdots \circ S_k \circ R \sqsubseteq R$, or $S_1 \circ \cdots \circ S_k \sqsubseteq R$ with $S_i \prec R$ for each i with $1 \le i \le k$.*

An RBox axiom *is an RIA. A* TBox axiom *(or* general concept inclusion *(GCI)) is an expression $C \sqsubseteq D$ where $C, D \in \mathsf{C}$. An* ABox axiom *is of the form $C(a)$ or $R(a, b)$ where $C \in \mathsf{C}$, $R \in \mathsf{R}$, and $a, b \in N_I$. A $\mathcal{SROIQV}(\mathcal{B}^s,\times)\mathcal{K}_{\mathcal{NF}}$* axiom *is any ABox, TBox, or RBox axiom, and a $\mathcal{SROIQV}(\mathcal{B}^s,\times)\mathcal{K}_{\mathcal{NF}}$* knowledge base *(KB) is a finite, regular set of $\mathcal{SROIQV}(\mathcal{B}^s,\times)\mathcal{K}_{\mathcal{NF}}$ axioms.*

Additionally $\mathcal{SROIQ}$ admits RBox axioms that directly express the empty role, role disjointness, asymmetry, reflexivity, irreflexibility, symmetry, and transitivity. As shown in [20], all these can be expressed in $\mathcal{SROIQV}(\mathcal{B}^s,\times)$ anyway, so we omit them here. In contrast to [20], we explicitly allow the usage of complex concepts and roles in the ABox to simplify the presentation in the next section. This difference is only syntactic, as such expressions can easily be reduced by introducing new concept and role names.

2.2 Semantics

The semantics of $\mathcal{SROIQV}(\mathcal{B}^s,\times)\mathcal{K}_{\mathcal{NF}}$ is a generalization of the semantics of $\mathcal{SROIQV}(\mathcal{B}^s,\times)$ [20] and that of $\mathcal{ALCK}_{\mathcal{NF}}$ [7]. We first recall two basic notions from [20].

An *interpretation* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consists of a domain $\Delta^\mathcal{I} \neq \emptyset$ and a function $\cdot^\mathcal{I}$ that maps elements in N_I, N_C, and N_R to elements, sets, and relations of $\Delta^\mathcal{I}$ respectively, i.e., for $a \in N_I$, $a^\mathcal{I} = d \in \Delta^\mathcal{I}$, for $A \in N_C$, $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, and, for $V \in N_R$, $V^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$. A *variable assignment* $\mathcal{Z}$ for an interpretation $\mathcal{I}$ is a function $\mathcal{Z} : N_V \to \Delta^\mathcal{I}$ such that, for each $v \in N_V$, $\mathcal{Z}(v) = a^\mathcal{I}$ for some $a \in N_I$.

As is common in MKNF-related semantics used to combine DLs with non-monotonic reasoning (see [7, 14, 16, 23]), specific restrictions on interpretations are introduced to ensure that certain unintended logical consequences can be avoided (see, e.g., [23]). We adapt the standard name assumption from [23].

Definition 3 *An interpretation $\mathcal{I}$ (over Σ to which $\approx$ is added) employs the* standard name assumption *if*

(1) N_I^* extends N_I with a countably infinite set of individuals that cannot be used in variable assignments, and $\Delta^\mathcal{I} = N_I^*$;
(2) for each i in N_I^*, $i^\mathcal{I} = i$; and
(3) equality $\approx$ is interpreted in $\mathcal{I}$ as a congruence relation – that is, $\approx$ is reflexive, symmetric, transitive, and allows for the replacement of equals by equals [8].

The first two conditions define $\mathcal{I}$ as a bijective function, while the third ensures that we still can identify elements of the domain.

It was shown in [23, Proposition 3.2] that we cannot distinguish between the consequences of first-order formulas under standard first-order semantics and under the standard name assumption. Therefore, we use the standard name assumption in the rest of the paper without referring to it further.

As an immediate side-effect, we note that the variable assignment is no longer tied to a specific interpretation. Similarly, we simplify notation by using Δ without reference to a concrete interpretation.

The first-order semantics is lifted to satisfaction in MKNF structures that treat the modal operators w.r.t. sets of interpretations.

Definition 4 *An* MKNF structure *is a triple $(\mathcal{I}, \mathcal{M}, \mathcal{N})$ where $\mathcal{I}$ is an interpretation, $\mathcal{M}$ and $\mathcal{N}$ are sets of interpretations, and $\mathcal{I}$ and all interpretations in $\mathcal{M}$ and $\mathcal{N}$ are defined over Δ. For any such $(\mathcal{I}, \mathcal{M}, \mathcal{N})$ and assignment $\mathcal{Z}$, the function $\cdot^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}$ is defined for arbitrary $\mathcal{SROIQV}(\mathcal{B}^s,\times)\mathcal{K}_{\mathcal{NF}}$ expressions as shown in Table 1.*

$(\mathcal{I}, \mathcal{M}, \mathcal{N})$ and $\mathcal{Z}$ satisfy *a $\mathcal{SROIQV}(\mathcal{B}^s,\times)\mathcal{K}_{\mathcal{NF}}$ axiom α, written $(\mathcal{I}, \mathcal{M}, \mathcal{N}), \mathcal{Z} \models \alpha$, if the corresponding condition in Table 1 holds. $(\mathcal{I}, \mathcal{M}, \mathcal{N})$* satisfies *$\alpha$, written $(\mathcal{I}, \mathcal{M}, \mathcal{N}) \models \alpha$, if $(\mathcal{I}, \mathcal{M}, \mathcal{N}), \mathcal{Z} \models \alpha$ for all variable assignments $\mathcal{Z}$. A (non-empty) set of interpretations $\mathcal{M}$* satisfies *α, written $\mathcal{M} \models \alpha$, if $(\mathcal{I}, \mathcal{M}, \mathcal{M}) \models \alpha$ holds for all $\mathcal{I} \in \mathcal{M}$. $\mathcal{M}$* satisfies *a $\mathcal{SROIQV}(\mathcal{B}^s,\times)\mathcal{K}_{\mathcal{NF}}$ knowledge base KB, written $\mathcal{M} \models KB$, if $\mathcal{M} \models \alpha$ for all axioms $\alpha \in KB$.*

Note the small deviations of the semantics of $\{t\}$, $\leqslant k\, S.C$, and $\geqslant k\, S.C$ in Table 1 compared to, e.g., that in [20]. These are necessary to ensure that the semantics of these three constructors works as intended under standard name assumption.

So far, we have extended the semantics of [20], considering not only an interpretation but also two sets of interpretations each of which is used to interpret one of the modal operators. We have also provided a monotonic semantics for $\mathcal{SROIQV}(\mathcal{B}^s, \times)\mathcal{K}_{\mathcal{NF}}$ KBs where the two sets used are identical (hence the interpretation of the two modal operators is exactly the same). Now, we will define a non-monotonic MKNF model in the usual fashion [7, 14, 16, 23]: $\mathcal{M}$ is fixed to interpret $\mathbf{A}$, and a superset $\mathcal{M}'$ is used to interpret $\mathbf{K}$ to test whether the knowledge derived from $\mathcal{M}$ is indeed minimal.

Table 1. Semantics of $\mathcal{SROIQV}(\mathcal{B}^s, \times)\mathcal{K}_{\mathcal{NF}}$

Syntax	Semantics
A	$A^{\mathcal{I}} \subseteq \Delta$
V	$V^{\mathcal{I}} \subseteq \Delta \times \Delta$
a	$a^{\mathcal{I}} \in \Delta$
x	$\mathcal{Z}(x) \in \Delta$
$\top$	Δ
$\bot$	$\emptyset$
$\{t\}$	$\{a \mid a \approx t^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}\}$
$\neg C$	$\Delta \setminus C^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}$
$C \sqcap D$	$C^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}} \cap D^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}$
$C \sqcup D$	$C^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}} \cup D^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}$
$\forall R.C$	$\{\delta \in \Delta \mid (\delta,\epsilon) \in R^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}$ implies $\epsilon \in C^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}\}$
$\exists R.C$	$\{\delta \in \Delta \mid \exists\epsilon$ with $(\delta,\epsilon) \in R^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}$ and $\epsilon \in C^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}\}$
$\exists S.\mathsf{Self}$	$\{\delta \in \Delta \mid (\delta,\delta) \in S^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}\}$
$\leqslant k\, S.C$	$\{\delta \in \Delta \mid \sharp\{([\delta]_{\approx},[\epsilon]_{\approx}) \in S^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}$ and $[\epsilon]_{\approx} \in C^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}\} \leq k\}$
$\geqslant k\, S.C$	$\{\delta \in \Delta \mid \sharp\{([\delta]_{\approx},[\epsilon]_{\approx}) \in S^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}$ and $[\epsilon]_{\approx} \in C^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}\} \geq k\}$
$\mathbf{K}C$	$\bigcap_{\mathcal{J}\in\mathcal{M}} C^{(\mathcal{J},\mathcal{M},\mathcal{N}),\mathcal{Z}}$
$\mathbf{A}C$	$\bigcap_{\mathcal{J}\in\mathcal{N}} C^{(\mathcal{J},\mathcal{M},\mathcal{N}),\mathcal{Z}}$
V^-	$\{(\delta,\epsilon) \in \Delta\times\Delta \mid (\epsilon,\delta) \in V^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}\}$
U	$\Delta \times \Delta$
$A \times B$	$\{(\delta,\epsilon) \in \Delta\times\Delta \mid \delta \in A^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}$ and $\epsilon \in B^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}\}$
$\neg S$	$(\Delta\times\Delta) \setminus S^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}$
$S_1 \sqcap S_2$	$S_1^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}} \cap S_2^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}$
$S_1 \sqcap S_2$	$S_1^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}} \cup S_2^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}$
$\mathbf{K}R$	$\bigcap_{\mathcal{J}\in\mathcal{M}} R^{(\mathcal{J},\mathcal{M},\mathcal{N}),\mathcal{Z}}$
$\mathbf{A}R$	$\bigcap_{\mathcal{J}\in\mathcal{N}} R^{(\mathcal{J},\mathcal{M},\mathcal{N}),\mathcal{Z}}$
$C(a)$	$a^{\mathcal{I}} \in C^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}$
$R(a,b)$	$(a^{\mathcal{I}},b^{\mathcal{I}}) \in R^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}$
$C \sqsubseteq D$	$C^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}} \subseteq D^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}$
$R_1 \circ \cdots \circ R_n \sqsubseteq R$	$R_1^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}} \circ \cdots \circ R_n^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}} \subseteq S^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}$

Interpretation $\mathcal{I}$; MKNF structure $(\mathcal{I},\mathcal{M},\mathcal{N})$; variable assignment $\mathcal{Z}$; $A, B \in N_C$; $C, D \in \mathsf{C}$; $V \in N_R$; $S_{(i)} \in \mathsf{R}^{\mathsf{s}}$;$R_{(i)} \in \mathsf{R}$;$a, b \in N_I$; $x \in N_V$, $t \in N_V \cup N_I$; $\circ$ composition of binary relations.

Definition 5 *Given a $\mathcal{SROIQV}(\mathcal{B}^s, \times)\mathcal{K}_{\mathcal{NF}}$ knowledge base KB, a (non-empty) set of interpretations $\mathcal{M}$ is an* MKNF model *of KB if (1) $\mathcal{M} \models KB$, and (2) for each $\mathcal{M}'$ with $\mathcal{M} \subset \mathcal{M}'$, $(\mathcal{I}', \mathcal{M}', \mathcal{M}) \not\models KB$ for some $\mathcal{I}' \in \mathcal{M}'$. KB is* MKNF-satisfiable *if an MKNF model of KB exists. An axiom α is* MKNF-entailed by *KB, written $KB \models_{\mathbf{K}} \alpha$, if all MKNF models $\mathcal{M}$ of KB satisfy α.*

As noted in [7], since $\mathcal{M} \models KB$ is defined w.r.t. $(\mathcal{I}, \mathcal{M}, \mathcal{M})$, the operators $\mathbf{K}$ and $\mathbf{A}$ are interpreted in the same way, and so we can restrict instance checking $KB \models_{\mathbf{K}} C(a)$ and subsumption $KB \models_{\mathbf{K}} C \sqsubseteq D$ to C and D without occurrences of the operator $\mathbf{A}$.

2.3 Decidability Considerations

In the following, we describe a decidable fragment of $\mathcal{SROIQV}(\mathcal{B}^s, \times)\mathcal{K}_{\mathcal{NF}}$ which in principle encompasses all the relevant other languages to be discussed in Section 4. Reasoning in this fragment follows in principle the approach from [7] for $\mathcal{ALCK}_{\mathcal{NF}}$ and its refinement from [14]: each model of a knowledge base in $\mathcal{SROIQV}(\mathcal{B}^s, \times)\mathcal{K}_{\mathcal{NF}}$ is cast into a $\mathcal{SROIQV}(\mathcal{B}^s, \times)$ KB. Consequently, reasoning in $\mathcal{SROIQV}(\mathcal{B}^s, \times)\mathcal{K}_{\mathcal{NF}}$ is reduced to a number of reasoning tasks in the non-modal $\mathcal{SROIQV}(\mathcal{B}^s, \times)$.

Following [20], we point out that we can simplify reasoning in $\mathcal{SROIQV}(\mathcal{B}^s, \times)\mathcal{K}_{\mathcal{NF}}$ to reasoning in $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ by grounding, i.e., by appropriately substituting nominal schemas by all nominals in all possible ways, and by simulating concept products as shown in [19, 25]. Note that neither grounding nor the material presented in the following are efficient for reasoning. But that does not constitute a problem since we only want to show decidability.

A set of interpretations $\mathcal{M}$ is *first-order representable* (alternatively, *$\mathcal{SROIQ}(\mathcal{B}^s)$ representable*) if there exists a first-order theory ($\mathcal{SROIQ}(\mathcal{B}^s)$ KB) $KB_{\mathcal{M}}$ such that $\mathcal{M} = \{\mathcal{I} \mid \mathcal{I}$ satisfies $KB_{\mathcal{M}}\}$. It is noted in [7] that, for $\mathcal{ALC}$, such a $KB_{\mathcal{M}}$ may be finite or infinite, and it is shown that, in general, models of $\mathcal{ALCK}_{\mathcal{NF}}$ KBs are not even first-order representable. Therefore the notion of subjectively quantified KBs is introduced in [7], and we extend this notion to $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KBs.

Building on the improved formalization in [14], we define that a $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ expression S is *subjective* if each $\mathcal{SROIQ}(\mathcal{B}^s)$ subexpression in S lies in the scope of at least one modal operator.

Definition 6 *A $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KB KB is* subjectively quantified *if each expression of the form $\exists R.C$, $\forall R.C$, $\leqslant k\, R.C$, and $\geqslant k\, R.C$ occurring in KB satisfies one of the conditions: R is a $\mathcal{SROIQ}(\mathcal{B}^s)$ role and C is a $\mathcal{SROIQ}(\mathcal{B}^s)$ concept, or R and C are both subjective.*

The overall idea is to avoid expressions that are only partially in scope of a modal operator. Besides not being first-order representable, such expressions yield counterintuitive consequences as shown in [7] (Section 3).

Following [7], we would now proceed to define a set of modal atoms, i.e., subjective expressions, appearing in such a subjectively quantified $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KB. This set can be partitioned into two sets of positive and negative modal atoms, i.e., the atoms that are assumed to hold and the ones that are assumed not to hold. A $\mathcal{SROIQ}(\mathcal{B}^s)$ representation of an MKNF model would be obtained from the modal atoms that are assumed to hold and the part of the considered KB that is free of modal operators. Instead, we restrict $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KBs even further. We use $\mathbf{M}$ to denote either $\mathbf{K}$ or $\mathbf{A}$, and $\mathbf{N}$ to denote either $\mathbf{M}$ or $\neg\mathbf{M}$.

Definition 7 *A $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KB KB is* strictly subjectively quantified *if the following conditions hold:*

1. *each expression of the form $\exists R.C$, $\forall R.C$, occurring in KB, satisfies one of two conditions: R is a $\mathcal{SROIQ}(\mathcal{B}^s)$ role and C is a $\mathcal{SROIQ}(\mathcal{B}^s)$ concept, or R is of the form $\mathbf{M}R'$ with $R' \in N_R$ and C is of the form $\mathbf{N}C'$ with C' a $\mathcal{SROIQ}(\mathcal{B}^s)$ concept;*
2. *modal operators are not allowed inside of statements of the form $\leqslant k\, R.C$, $\geqslant k\, R.C$, and $\exists S.\mathsf{Self}$ nor in RIAs;*
3. *for assertions $R(a, b) \in KB$, we have either $R \in N_R$ or R is of the form $\mathbf{M}R_1$ and $R_1 \in N_R$.*

This restriction is severe, but it approximates the conditions to obtain subjectively quantified $\mathcal{ALCK}_{\mathcal{NF}}$ KBs in [7], which significantly simplifies the following steps outlined before Definition 7.

From now on, we follow closely the argument in [7]. We consider two disjoint subsets of the $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ TBox $\mathcal{T}$ of KB: $\mathcal{T}'$, the set of all axioms that contain no modal operators, and Γ, the set of all axioms that contain at least one modal operator. Moreover, we say that D *occurs strictly* in C if there is an occurrence of D in C which lies outside the scope of both modal operators and quantifiers.

Given domain Δ and $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KB KB, the set of *modal atoms* $MA_\Delta(KB)$ is defined inductively as follows:

(1) if $C(a) \in KB$, then $\mathbf{K}C(a) \in MA_\Delta(KB)$;
(2) if $R(a, b) \in KB$, then $\mathbf{K}R(a, b) \in MA_\Delta(KB)$;
(3) if $\mathbf{M}R(a, b) \in KB$, then $\mathbf{M}R(a, b) \in MA_\Delta(KB)$;
(4) if $\mathbf{K}C \sqsubseteq D \in \Gamma$, then $\mathbf{K}C(x), \mathbf{K}D(x) \in MA_\Delta(KB)$ for each $x \in \Delta$;
(5) if $\mathbf{M}D$ occurs strictly in C, and there exists a modal atom $\mathbf{M}'C(x) \in MA_\Delta(KB)$, then $\mathbf{M}D(x) \in MA_\Delta(KB)$;
(6) if $\exists \mathbf{M}_1 R.\mathbf{M}_2 D$ (resp. $\exists \mathbf{M}_1 R.\neg\mathbf{M}_2 D$) occurs strictly in C and there exists a modal atom $\mathbf{M}'C(x) \in MA_\Delta(KB)$, then $\exists \mathbf{M}_1 R.\mathbf{M}_2 D(x) \in MA_\Delta(KB)$ ($\exists \mathbf{M}_1 R.\neg\mathbf{M}_2 D(x) \in MA_\Delta(KB)$) and $\mathbf{M}_1 R(x, y), \mathbf{M}_2 D(y) \in MA_\Delta(KB)$ for each $y \in \Delta$;
(7) if $\forall \mathbf{M}_1 R.\mathbf{M}_2 D$ (resp. $\forall \mathbf{M}_1 R.\neg\mathbf{M}_2 D$) occurs strictly in C and there exists a modal atom $\mathbf{M}'C(x) \in MA_\Delta(KB)$, then $\forall \mathbf{M}_1 R.\mathbf{M}_2 D(x) \in MA_\Delta(KB)$ ($\forall \mathbf{M}_1 R.\neg\mathbf{M}_2 D(x) \in MA_\Delta(KB)$) and $\mathbf{M}_1 R(x, y), \mathbf{M}_2 D(y) \in MA_\Delta(KB)$ for each $y \in \Delta$;
(8) nothing else belongs to $MA_\Delta(KB)$.

Note that axioms not corresponding to (4) can easily be rewritten, e.g., $\mathbf{K}C \sqcap \mathbf{K}C' \sqsubseteq D$ becomes $\mathbf{K}C \sqsubseteq \neg\mathbf{K}C' \sqcup D$. Moreover, since Δ is infinite, $MA_\Delta(KB)$ is in general infinite because of (4), (6), and (7).

A *partition* of $MA_\Delta(KB)$ is a pair (P, N) of *positive modal atoms* P and *negative modal atoms* N such that $P \cap N = \emptyset$ and $P \cup N = MA_\Delta(KB)$. $C(x)(P, N)$ denotes the assertion obtained from $C(x)$ by:

- replacing each strict occurrence of a concept expression $\mathbf{M}D$ in C with $\top$ if $\mathbf{M}D(x) \in P$ and with $\bot$ otherwise;
- replacing each strict occurrence of $\exists \mathbf{M}R.\mathbf{N}D$ in C with $\top$ if $\exists \mathbf{M}R.\mathbf{N}D(x) \in P$ and with $\bot$ otherwise;
- replacing each strict occurrence of $\forall \mathbf{M}R.\mathbf{N}D$ in C with $\top$ if $\forall \mathbf{M}R.\mathbf{N}D(x) \in P$ and with $\bot$ otherwise.

This allows us to define the objective knowledge. Let KB be a $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KB, (P, N) a partition of $MA_\Delta(KB)$, $\mathcal{T}'$ the TBox axioms of KB which are free of modal operators, and $\mathcal{R}$ the set of RBox axioms of KB. We denote with $Ob_K(P, N)$ and $Ob_A(P, N)$ the following $\mathcal{SROIQ}(\mathcal{B}^s)$ KBs:

$$\begin{aligned} Ob_K(P, N) &= \mathcal{T}' \cup \mathcal{R} \cup \{C(x)(P, N) \mid \mathbf{K}C(x) \in P\} \\ &\quad \cup \{R(x, y)(P, N) \mid \mathbf{K}R(x, y) \in P\} \\ Ob_A(P, N) &= \mathcal{T}' \cup \mathcal{R} \cup \{C(x)(P, N) \mid \mathbf{A}C(x) \in P\} \\ &\quad \cup \{R(x, y)(P, N) \mid \mathbf{A}R(x, y) \in P\} \end{aligned}$$

Here, we slightly abuse notation, since the assertions do not exactly correspond to $\mathcal{SROIQ}(\mathcal{B}^s)$ ABox axioms as defined.

We say that a partition (P, N) of $MA_\Delta(KB)$ is *consistent* with KB if the following conditions hold:

(1) if $C(a) \in KB$, then $\mathbf{K}C(a) \in P$ and, if $R(a, b) \in KB$, then $\mathbf{K}R(a, b) \in P$;
(2) the $\mathcal{SROIQ}(\mathcal{B}^s)$ KB $Ob_K(P, N)$ is satisfiable;
(3) the $\mathcal{SROIQ}(\mathcal{B}^s)$ KB $Ob_A(P, N)$ is satisfiable;
(4) $Ob_K(P, N) \not\models C(x)(P, N)$ for each $\mathbf{K}C(x) \in N$;
(5) $Ob_K(P, N) \not\models R(x, y)$ for each $\mathbf{K}R(x, y) \in N$;
(6) $Ob_A(P, N) \not\models C(x)(P, N)$ for each $\mathbf{A}C(x) \in N$;
(7) $Ob_A(P, N) \not\models R(x, y)$ for each $\mathbf{A}R(x, y) \in N$;
(8) for each $\exists \mathbf{M}_1 R.\mathbf{M}_2 D(x) \in P$ ($\exists \mathbf{M}_1 R.\neg\mathbf{M}_2 D(x) \in P$), there exists y such that $\mathbf{M}_1 R(x, y) \in P$ and $\mathbf{M}_2 D(y) \in P$ ($\mathbf{M}_2 D(y) \in N$);
(9) for each $\forall \mathbf{M}_1 R.\mathbf{M}_2 D(x) \in P$ ($\forall \mathbf{M}_1 R.\neg\mathbf{M}_2 D(x) \in P$) and for each y such that $\mathbf{M}_1 R(x, y) \in P$, $\mathbf{M}_2 D(y) \in P$ ($\mathbf{M}_2 D(y) \in N$);
(10) for each $\mathbf{K}C \sqsubseteq D \in \Gamma$ and for each $x \in \Delta$, if $\mathbf{K}C(x) \in P$, then $\mathbf{K}D(x) \in P$.

As pointed out in [7], if $MA_\Delta(KB)$ is infinite, then verifying whether (P, N) is consistent requires to check infinitely many conditions. Intuitively, a guessed partition (P, N) corresponds to the (infinite) $\mathcal{SROIQ}(\mathcal{B}^s)$ KB $Ob_K(P, N)$. If (P, N) is consistent with KB, then a candidate model for KB is obtained from all the interpretations that satisfy $Ob_K(P, N)$. A link between pairs of interpretations and partitions is established in the following. For that purpose we lift satisfaction from a set of interpretations (Definition 2) to a pair of sets of interpretations: $(\mathcal{M}, \mathcal{N}) \models \alpha$ holds if $(\mathcal{I}, \mathcal{M}, \mathcal{N}) \models \alpha$ holds for all $\mathcal{I} \in \mathcal{M}$.

Let $(\mathcal{M}_1, \mathcal{M}_2)$ be a pair of of sets of interpretations over Δ. We say that $(\mathcal{M}_1, \mathcal{M}_2)$ *induces* the following partition (P, N) of $MA_\Delta(KB)$:

$$\begin{aligned} P = &\{C(x) \mid C(x) \in MA_\Delta(KB) \text{ and } (\mathcal{M}_1, \mathcal{M}_2) \models C(x)\} \\ &\cup \{\mathbf{M}R(x, y) \mid \mathbf{M}R(x, y) \in MA_\Delta(KB) \text{ and} \\ &(\mathcal{M}_1, \mathcal{M}_2) \models \mathbf{M}R(x, y)\} \\ N = &\{C(x) \mid C(x) \in MA_\Delta(KB) \text{ and } (\mathcal{M}_1, \mathcal{M}_2) \not\models C(x)\} \\ &\cup \{\mathbf{M}R(x, y) \mid \mathbf{M}R(x, y) \in MA_\Delta(KB) \text{ and} \\ &(\mathcal{M}_1, \mathcal{M}_2) \not\models \mathbf{M}R(x, y)\} \end{aligned}$$

We can show that the intended correspondence indeed holds.

Lemma 1 *Let $(\mathcal{M}_1, \mathcal{M}_2)$ be a pair of sets of interpretations over Δ such that $\mathcal{M}_1 \supseteq \mathcal{M}_2$ and $(\mathcal{M}_1, \mathcal{M}_2)$ satisfies KB, and let (P, N) be the partition of $MA_\Delta(KB)$ induced by $(\mathcal{M}_1, \mathcal{M}_2)$. Then (P, N) is consistent with KB.*

Based on that, it can be shown that there exists a one-to-one correspondence between every model $\mathcal{M}$ of KB and the partition induced by $(\mathcal{M}, \mathcal{M})$.

Theorem 1 *A set $\mathcal{M}$ of interpretations over Δ is an MKNF model for $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KB KB iff the partition of $MA_\Delta(KB)$ induced by $(\mathcal{M}, \mathcal{M})$ satisfies the following conditions:*

(1) (P, N) *is consistent with* KB*;*
(2) $\mathcal{M} = \{\mathcal{I} \mid \mathcal{I} \models Ob_K(P, N)\}$*;*
(3) for each $\mathbf{A}C(x) \in N$*,* $Ob_K(P, N) \not\models C(x)(P, N)$*, and for each* $\mathbf{A}R(x, y) \in N$*,* $Ob_K(P, N) \not\models R(x, y)$*;*
(4) $Ob_K(P, N) \models Ob_A(P, N)$*;*
(5) for each partition (P', N') *of* $MA_\Delta(KB')$*, where* $KB' = KB \cup \{\mathbf{A}C(x) \mid C(x) \in Ob_K(P, N)\} \cup \{\mathbf{A}R(x, y) \mid R(x, y) \in Ob_K(P, N)\}$*, at least one of the following conditions does not hold:*
   (a) (P', N') *is consistent with* KB'*;*
   (b) $Ob_K(P, N) \models Ob_K(P', N')$*;*
   (c) $Ob_K(P', N') \not\models Ob_K(P, N)$*;*
   (d) $Ob_K(P, N) \models Ob_A(P', N')$*.*

As an immediate consequence, we obtain that the models of a strictly subjectively quantified $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KB KB are $\mathcal{SROIQ}(\mathcal{B}^s)$-representable.

Corollary 1 *Let KB be a strictly subjectively quantified $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KB, $\mathcal{M}$ an MKNF model of KB, and (P, N) be the partition of $MA_\Delta(KB)$ induced by $(\mathcal{M}, \mathcal{M})$. Then $\mathcal{M} = \{\mathcal{I} \mid \mathcal{I} \models Ob_K(P, N)\}$.*

This does not yield a decidable procedure yet, since subjectively quantified $\mathcal{ALCK}_{\mathcal{NF}}$ KBs may have an infinite representation or an infinite number of models [7] and the same holds for strictly subjectively quantified $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KBs. To counter that, a further restriction is introduced in [7].

Definition 8 *Let KB be a $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KB that is strictly subjectively quantified, $\mathcal{A}$ its ABox, and Γ the set of axioms in the TBox of KB that contain at least one modal operator. A concept C in KB is* simple *if, for all expressions of the form $\exists\mathbf{A}R.\mathbf{N}D$ or $\forall\mathbf{A}R.\mathbf{N}D$ in C, D has no occurrence of role expressions of the form $\mathbf{K}R$. KB is* simple *if the following conditions are satisfied:*

1. *only axioms of the form $\mathbf{K}C \sqsubseteq D$ occur in Γ, where C is a $\mathcal{SROIQ}(\mathcal{B}^s)$ concept and no $\mathbf{K}$ operator occurs in $\exists$ and $\forall$ restrictions in D;*
2. *for each $\mathbf{K}C \sqsubseteq D \in \Gamma$, $(KB \setminus (\Gamma \cup \mathcal{A})) \not\models \top \sqsubseteq C$;*
3. *all concept expressions in $\mathcal{A}$ are simple.*

Considering these restrictions, we can show that a decidable reasoning procedure exists.

Theorem 2 *Let KB be a simple $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KB. Then the MKNF models $\mathcal{M}$ of KB can be characterized by a finite subset of $MA_\Delta(KB)$.*

Based on partitions (P, N) of the modal atoms $MA_\Delta(KB)$ appearing in a strictly subjectively quantified $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KB KB, the notion 'induces', and $Ob_K(P, N)$, a $\mathcal{SROIQ}(\mathcal{B}^s)$ representation of the models of KB, it can be shown that the models of KB are $\mathcal{SROIQ}(\mathcal{B}^s)$-representable (for details, also w.r.t. the just mentioned notions, we refer to the extended version and also [7]).

Corollary 2 *Let KB be a strictly subjectively quantified $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KB, $\mathcal{M}$ an MKNF model of KB, and (P, N) be the partition of $MA_\Delta(KB)$ induced by $(\mathcal{M}, \mathcal{M})$. Then $\mathcal{M} = \{\mathcal{I} \mid \mathcal{I} \models Ob_K(P, N)\}$.*

3 Examples

In this section we present some modeling examples illustrating the characteristics of $\mathcal{SROIQV}(\mathcal{B}^s, \times)\mathcal{K}_{\mathcal{NF}}$. It must be kept in mind, however, that the main purpose of this research is to advance the quest for a unifying logic for the Semantic Web stack. We do not intend to suggest at this stage that our proposed language be used as a language in practical applications.

Let us look at the introductory example from [20]:

$$\text{HasParent}(\text{mary}, \text{john}) \tag{1}$$

$$(\exists\text{HasParent}.\exists\text{Married}.\{\text{john}\})(\text{mary}) \tag{2}$$

$$\exists\text{HasParent}.\{z\} \sqcap \exists\text{HasParent}.\exists\text{Married}.\{z\} \sqsubseteq C \tag{3}$$

Axiom (3) in fact uses nominal schemas to express a DL-safe rule [20], and it allows us to conclude that $C(\text{mary})$ holds despite the fact that we do not know who precisely the other parent is. If we interpret this example under our new semantics, we retain all logical consequences. The only thing to point out is that the unique MKNF model consists of all (first-order) interpretations that satisfy the KB.

Consider substituting (3) with the following axiom.

$$\mathbf{K}\exists\text{HasParent}.\{z\} \sqcap \mathbf{K}\exists\text{HasParent}.\exists\text{Married}.\{z\} \sqsubseteq \mathbf{K}C$$

If we do this, nothing changes in terms of consequences, i.e., we can still derive $C(\text{mary})$, which corresponds to $\mathbf{K}C(\text{mary})$. It is remarkable that we are able to reason over unknown individuals in MKNF in this way, simply because the unknown individual is hidden and not relevant for the intersections over all interpretations in an MKNF model.

Discussing the example in more detail, any MKNF model $\mathcal{M}$ of the new axiom plus (1) and (2) must ensure that each $\mathcal{I} \in \mathcal{M}$ satisfies (1) and (2), which means that, e.g., $(\text{mary}, \text{john}) \in \text{HasParent}^{\mathcal{I}}$ for each such $\mathcal{I}$. As for the substitution of (3), on any assignment, if $z = \text{mary}$, then the left hand side is simply empty (Mary is not known to be a parent of anyone). Thus, in this case the GCI does not impose anything. If $z = \text{john}$, then the intersection over all $\mathcal{I} \in \mathcal{M}$ of $(\exists\text{HasParent}.\{\text{john}\})^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}$ contains only mary. The same holds for the second conjunct, and so for all $\mathcal{I} \in \mathcal{M}$, it must be that $\text{mary} \in C^{\mathcal{I}}$. That is of course the only consequence, so the MKNF model contains all first-order interpretations that model (1), (2), and $C(\text{mary})$.

Now consider replacing (3) with the following.

$$\mathbf{K}\exists\text{HasParent}.\{z\} \sqcap \mathbf{K}\exists\text{HasParent}.\exists\text{Married}.\{z\} \sqsubseteq \mathbf{A}C$$

The effect of this is that the KB is not satisfiable, since the above axiom corresponds to an integrity constraint. $\mathcal{M}$ is used for both $\mathbf{K}$ and $\mathbf{A}$ when checking for satisfiability. Hence the argument from above applies again, and we are required to have $C(\text{mary})$ in every $\mathcal{I} \in \mathcal{M}$. Keeping $\mathcal{M}$ for the evaluation of $\mathbf{A}$ but using any $\mathcal{M}'$ with one $\mathcal{I}'$ such that $C(\text{mary}) \notin \mathcal{I}'$, then $(\mathcal{I}', \mathcal{M}', \mathcal{M}) \models KB$. I.e., $\mathcal{M}$ is not an MKNF model. Note that there is no other set $\mathcal{M}$ such that $(\mathcal{I}, \mathcal{M}, \mathcal{M}) \models KB$. Note also that if $C(\text{mary})$ were added as a fact, then $\mathcal{M}$ would be a model, because $\mathcal{M}'$ would not satisfy that fact.

In other words, $\mathbf{K}\varphi$ on the right hand side enforces actively that φ is known, while $\mathbf{A}\varphi$ only tests whether φ is known in the sense of a constraint.

Modal operators can appear nested in DL expressions, with some interesting effects.

$$\exists\text{HasParent}.\{z\} \sqcap \mathbf{K}\exists\mathbf{K}\text{HasParent}.\mathbf{K}\exists\mathbf{K}\text{Married}.\mathbf{K}\{z\} \sqsubseteq C$$

In the second conjunct, the first $\mathbf{K}$ applies to the entire conjunct as before. The second applies directly to HasParent and requires that one pair of individuals appears in that relation in all interpretations of an MKNF model $\mathcal{M}$. The third asks for an individual i such that $\exists\mathbf{K}\text{Married}.\mathbf{K}\{z\}(i)$ holds for all $\mathcal{I}$ in an MKNF model. The fourth applies directly to the relation Married and asks for a known pair in Married (in all $\mathcal{I} \in \mathcal{M}$). The final one asks for a known $\{z\}$, i.e., the only potential possibilities for $\{z\}$ are {john} and {mary}. Note that we can no longer derive $C(mary)$ simply because we do not know who the other parent is, meaning that in different $\mathcal{I} \in \mathcal{M}$, different unknown individuals are used for this parent.

Default negation can be introduced via $\neg\mathbf{A}$.

$$\exists\text{HasParent}.\{z\} \sqcap \exists\text{HasParent}.\exists\neg\mathbf{A}\text{Married}.\{z\} \sqsubseteq C$$

The intention above is that C is the class of all individuals whose parents are not (known to be) married. However, the axiom does not have the desired effect here, because there is no known individual that matches as second parent. So $\mathbf{A}$Married is empty and $\neg\mathbf{A}$Married is $\top$. Consequently, C(mary) holds, even though we know that Mary's parents are married.[5]

The very idea is clearly expressible in hybrid MKNF, and thus expressible in $\mathcal{SROIQV}(\mathcal{B}^s, \times)\mathcal{K}_{\mathcal{NF}}$ (see Section 4.2 below). It just has to be noted that in a hybrid MKNF formalization *all* variables are DL-safe and not just some of them. In fact, the above example still works as long as the individuals are known. Assume that the second parent is known by name as well. Depending on whether the marriage between that person and john is known or not, we obtain C(mary) only if the marriage is unknown.

In any case, our example indicates that we can reason over unknown individuals and use DL-safe default negation safely within one axiom. But a mixing of these within one subexpression may have unexpected consequences.

4 Coverage of Other Languages

We now turn to the key results of our proposal, namely that $\mathcal{SROIQV}(\mathcal{B}^s, \times)\mathcal{K}_{\mathcal{NF}}$ encompasses some of the most prominent languages related to OWL, rules, non-monotonic reasoning, and their integrations. Some results follow trivially from the definitions in Section 2 and from previous work. Some work is needed to show coverage of n-ary Datalog (Section 4.1) and Hybrid MKNF (Section 4.2). $\mathcal{SROIQV}(\mathcal{B}^s, \times)\mathcal{K}_{\mathcal{NF}}$ then encompasses the following.

- $\mathcal{SROIQ}$ (a.k.a. OWL 2 DL).
- The tractable profiles OWL 2 EL, OWL 2 RL, OWL 2 QL.
- RIF-Core [3], i.e., n-ary Datalog, interpreted as DL-safe Rules [24]. Coverage of binary Datalog is shown in [20], while the general case is shown in Section 4.1.
- DL-safe SWRL [24], $\mathcal{AL}$-log [6], and CARIN [21]. Coverage follows from Section 4.1 (alternatively from Section 4.2 in conjunction with [23]).
- $\mathcal{ALCK}_{\mathcal{NF}}$. This follows from our definitions and includes notions of concept and role closure present in this formalism.
- Closed Reiter defaults, a form of non-monotonic reasoning, are covered through the coverage of $\mathcal{ALCK}_{\mathcal{NF}}$. This includes coverage of DLs extended with default rules as presented in [2].
- Hybrid MKNF (see Section 4.2 below).
- Answer Set Programming [9], i.e., disjunctive Datalog with classical negation and non-monotonic negation under the answer set semantics. This follows from the coverage of Hybrid MKNF [23].

[5] This argument holds irrespective of the fact that John is a known individual who is not known to be married to itself.

4.1 n-ary Datalog

Below, we generalize a result found in [20] on embedding DL-safe rules into $\mathcal{SROIQV}(\mathcal{B}^s, \times)$. Specifically, we extend the result to apply to rules in which predicates of arbitrary arity appear. Some notation and definitions are also adopted from [20].

In the following, RB is a set of Datalog rules defined over a signature $\Sigma = \langle N_I, N_P, N_V\rangle$, where N_I, N_P, and N_V, are sets of constants, n-ary predicates, and variables, respectively. Each rule has the form $A_1, \ldots, A_n \rightarrow H$, where H and each A_i is of the form $P(t_1, \ldots, t_n)$, with $P \in N_P$ and each $t_i \in N_I \cup N_V$. $N_{P,i}$ ($N_{P,>i}$) is the set of predicates of N_P with arity i (greater than i). Also, $\top, \bot \in N_{P,1}$, and $\approx \in N_{P,2}$.

Definition 9 *Interpretations $\mathcal{I}$ and variable assignments $\mathcal{Z}$ are defined as in Section 2.2 under the standard name assumption. The function $\cdot^{(\mathcal{I},\mathcal{M},\mathcal{N}),\mathcal{Z}}$ from Table 1 is generalized to n-ary predicates by assigning a relation $P^{\mathcal{I}} \subseteq \Delta^n$ to each $P \in N_P$ with $n > 2$.*

Since Datalog rules are free of modal operators, we need only refer to $\mathcal{I}, \mathcal{Z}$. An atom $P(t_1, \ldots, t_n)$ is satisfied *by $\mathcal{I}$ and $\mathcal{Z}$, written $\mathcal{I}, \mathcal{Z} \models P(t_1, \ldots, t_n)$ if $(t_1^{\mathcal{I},\mathcal{Z}}, \ldots, t_n^{\mathcal{I},\mathcal{Z}}) \in P^{\mathcal{I}}$. A set of atoms $\mathcal{B}$ is* satisfied *by $\mathcal{I}$ and $\mathcal{Z}$ ($\mathcal{I}, \mathcal{Z} \models \mathcal{B}$) if $\mathcal{I}, \mathcal{Z} \models A_i$ for each $A_i \in \mathcal{B}$. A rule $\mathcal{B} \rightarrow H$ is* satisfied *by $\mathcal{I}$ and $\mathcal{Z}$ ($\mathcal{I}, \mathcal{Z} \models \mathcal{B} \rightarrow H$) if $\mathcal{I}, \mathcal{Z} \models H$ or $\mathcal{I}, \mathcal{Z} \not\models \mathcal{B}$. $\mathcal{I}$* satisfies *a rule $\mathcal{B} \rightarrow H$ if, for all assignments $\mathcal{Z}$, $\mathcal{I}, \mathcal{Z} \models \mathcal{B} \rightarrow H$. $\mathcal{I}$ satisfies RB if it satisfies all $r \in RB$.*

DL-safe rules with n-nary predicates can be embedded into an equisatisfiable $\mathcal{SROIQV}(\mathcal{B}^s, \times)$ KB $\mathsf{dl}(RB)$ over the signature $\Sigma = \langle N_I, N_C, N_R, N_V\rangle$. Here, $N_C = N_{P,1}$ and $N_R = N_{P,2} \cup \{U\} \cup S$, where S is a special set of roles: If $P \in N_{P,>2}$ has arity k, then $P_1, \ldots, P_k \in S$ are unique binary predicates associated with P; S is the set of all such predicates.

The rules defining $\mathsf{dl}(RB)$ are shown below. C and R are unary and binary predicates in RB, while P has higher arity.

1. $\mathsf{dl}(C(t)) := \exists U.(\{t\} \sqcap C)$;
2. $\mathsf{dl}(R(t, u)) := \exists U.(\{t\} \sqcap \exists R.\{u\})$;
3. $\mathsf{dl}(P(t_1, \ldots, t_k)) := \exists U.(\exists P_1.\{t_1\} \sqcap \ldots \sqcap \exists P_k.\{t_k\})$;
4. $\mathsf{dl}(A_1, \ldots, A_n \rightarrow H) := \mathsf{dl}(A_1) \sqcap \ldots \sqcap \mathsf{dl}(A_n) \sqsubseteq \mathsf{dl}(H)$;
5. $\mathsf{dl}(RB)$:=$\{\mathsf{dl}(r) | r \in RB\}$.

Definition 10 *Given a set of interpretations $\mathcal{M}$ for RB, consider one $\mathcal{I} \in \mathcal{M}$. We define a family fam($\mathcal{I}$) of interpretations $\mathcal{J}$.*

(a) To each $(d_1, \ldots, d_k) \in P^{\mathcal{I}}$, assign a unique element e in Δ (i.e., we define a total, injective function from the set of tuples to Δ).
(b) For each $C \in N_{P,1}$, $C^{\mathcal{J}} := C^{\mathcal{I}}$.
(c) For each $R \in N_{P,2}$, $R^{\mathcal{J}} := R^{\mathcal{I}}$.
(d) For each $P \in N_{P,>2}$, if $(d_1, \ldots, d_k) \in P^{\mathcal{I}}$, then $(e, d_i) \in P_i^{\mathcal{J}}$, where e is the element assigned to $(d_1, \ldots, d_k)$ in point (a).

The standard name assumption applies, so the domain is always the same, and elements in N_I are always mapped to the same $d \in \Delta$.

Any interpretation $\mathcal{J}$ for $\mathsf{dl}(RB)$ can be reduced to an interpretation $\mathcal{I}$ for RB: if $(e, d_1) \in P_1^{\mathcal{J}}$, ..., $(e, d_k) \in P_k^{\mathcal{J}}$, then $(d_1, \ldots, d_k) \in P^{\mathcal{I}}$. And so, for any interpretation $\mathcal{J}$ for $\mathsf{dl}(RB)$, there is an $\mathcal{I}$ for RB such that $\mathcal{J} \in$ fam($\mathcal{I}$).

Lemma 2 *Let A be an atom in RB, $\mathcal{I}$ an interpretation of RB, $\mathcal{J} \in$ fam($\mathcal{I}$), and $\mathcal{Z}$ a variable assignment.*

1. $\mathcal{I}, \mathcal{Z} \models A$ if and only if $\mathsf{dl}(A)^{\mathcal{J},\mathcal{Z}} = \Delta$. and
2. $\mathcal{I}, \mathcal{Z} \not\models A$ if and only if $\mathsf{dl}(A)^{\mathcal{J},\mathcal{Z}} = \emptyset$.

Theorem 3 *Let RB be a Datalog program. $\mathcal{M}$ is the set of all interpretations $\mathcal{I}$ that satisfy RB if and only if $\mathcal{M}_1 = \{\mathcal{J} \mid \mathcal{J} \in fam(\mathcal{I})$ with $\mathcal{I} \in \mathcal{M}\}$ is the set of all interpretations that satisfy $\mathsf{dl}(RB)$.*

It also follows immediately from Theorem 3 that the tractable language $\mathcal{SROELV}_n$ introduced in [20] encompasses not only $\mathcal{SROEL}$ a.k.a. OWL 2 EL, but also n-ary Datalog.

4.2 Hybrid MKNF Knowledge Bases

Hybrid MKNF knowledge bases [23], which are based on MKNF logics [22], i.e., first-order logic with equality plus two modal operators $\mathbf{K}$ and $\mathbf{not}$, are defined as the combination of a decidable DL knowledge base and a set of rules.

Given a DL knowledge base $\mathcal{O}$, a (function-free) first-order atom $P(t_1, \ldots, t_n)$ is a DL-atom if P is $\approx$ or is in $\mathcal{O}$; otherwise it is a non-DL-atom. An *MKNF rule* r has the below form, where H_k, A_i, and B_j are (possibly classically negated) first-order atoms:

$$\mathbf{K}H_1 \vee \mathbf{K}H_l \leftarrow \mathbf{K}A_1, \ldots, \mathbf{K}A_n, \mathbf{not}B_1, \ldots, \mathbf{not}B_m \qquad (4)$$

A program $\mathcal{P}$ is a finite set of MKNF rules, and a *hybrid MKNF knowledge base* $\mathcal{K}$ is a pair $(\mathcal{O}, \mathcal{P})$. The *ground instantiation* of $\mathcal{K}$ is the KB $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ where $\mathcal{P}_G$ is obtained from $\mathcal{P}$ by replacing each rule r of $\mathcal{P}$ with a set of rules substituting each variable in r with constants from $\mathcal{K}$ in all possible ways.

In [23], MKNF KBs are considered in which H_k, A_i, and B_j may be arbitrary first-order formulas. Here, the restriction to (classically negated) first-order atoms suffices and simplifies the presentation.

Hybrid MKNF KBs are embedded into MKNF logics. We briefly recall the syntax and semantics of (function-free) MKNF logics.

Let $\Sigma = (N_I, N_P, N_V)$ be a signature and N_P contain the equality predicate $\approx$. The syntax of MKNF formulas over Σ is defined by the below grammar, where $t_i \in N_I \cup N_V$ and $P \in N_P$.

$$\varphi \leftarrow P(t_1, \ldots, t_n) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x.\varphi \mid \mathbf{K}\varphi \mid \mathbf{not}\varphi \qquad (5)$$

Moreover, $\varphi_1 \vee \varphi_2$, $\varphi_1 \supset \varphi_2$, $\varphi_1 \equiv \varphi_2$, $\forall x : \varphi$, $\top$, $\bot$, $t_1 \approx t_2$, and $t_1 \not\approx t_2$ are admitted standard syntactic shortcuts.

First-order atoms of the form $t_1 \approx t_2$ (resp. $t_1 \not\approx t_2$) are called *equalities* (resp. *inequalities*), and $\varphi[t_1/x_1, \ldots, t_n/x_n]$ denotes the formula obtained by substituting the free variables x_i in φ, i.e., those that are not in the scope of any quantifier, by the terms t_i. φ is closed if it contains no free variables. Given a (first-order) formula φ, $\mathbf{K}\varphi$ is called a $\mathbf{K}$-*atom* and $\mathbf{not}\varphi$ a $\mathbf{not}$-*atom*; $\mathbf{K}$-atoms and $\mathbf{not}$-atoms are *modal atoms*. As in n-nary Datalog, N_P contains N_C and N_R. The generalization of interpretations to n-ary $P \in N_P$ also applies.

Let φ be a closed MKNF formula. Given an MKNF structure $(\mathcal{I}, \mathcal{M}, \mathcal{N})$, satisfaction of φ is defined as in [23, Table II]. We say that a set of interpretations $\mathcal{M}$ *satisfies* φ, written $\mathcal{M} \models \varphi$, if $(\mathcal{I}, \mathcal{M}, \mathcal{M}) \models \varphi$ for each $\mathcal{I} \in \mathcal{M}$.

A set of interpretations $\mathcal{M}$ is an *MKNF model* of φ if (1) $\mathcal{M}$ satisfies φ, and (2) for each set of interpretations $\mathcal{M}'$ with $\mathcal{M}' \supset \mathcal{M}$ we have $(\mathcal{I}', \mathcal{M}', \mathcal{M}) \not\models \varphi$ for some $\mathcal{I}' \in \mathcal{M}'$.

An MKNF formula φ is *MKNF-satisfiable* if an MKNF model of φ exists. Furthermore, φ *MKNF-entails* ψ, written $\varphi \models_{\mathbf{K}} \psi$, if $\mathcal{M} \models \psi$ for each MKNF model $\mathcal{M}$ of φ.

The definition above is similar to Definition 5, only that in the earlier definition, sets of (pairs of) individuals are considered, while here the satisfaction relation from [23, Table II] is used. We recall the embedding of hybrid MKNF KBs into MKNF logics.

Let $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ be a hybrid MKNF knowledge base and $\pi(\mathcal{O})$ the transformation of $\mathcal{O}$ into a formula of first-order logic with equality. We extend π to MKNF rules r of the form (4), $\mathcal{P}$, and $\mathcal{K}$ as follows, where $\vec{x}$ is the vector of the free variables of r.

$$\pi(r) = \forall \vec{x} : (\mathbf{K}A_1 \wedge \ldots \wedge \mathbf{K}A_n \wedge \mathbf{not}B_1 \wedge \ldots \wedge \mathbf{not}B_m \supset \mathbf{K}H_1 \vee \ldots \vee \mathbf{K}H_l)$$

$$\pi(\mathcal{P}) = \bigwedge_{r \in \mathcal{P}} \pi(r) \qquad \pi(\mathcal{K}) = \mathbf{K}\pi(\mathcal{O}) \wedge \pi(\mathcal{P})$$

We abuse notation and use $\mathcal{K}$ instead of $\pi(\mathcal{K})$. The following syntactic restriction, similar in spirit to the restriction applied to nominal schemas, ensures decidability. An MKNF rule r is *DL-safe* if every variable in r occurs in at least one non-DL-atom $\mathbf{K}A_i$ in the body of r. $\mathcal{K}$ is *DL-safe* if all the rules in $\mathcal{K}$ are DL-safe.

As argued in [23], reasoning in hybrid MKNF can thus be restricted to ground $\mathcal{K}_G$. We thus use the variable assignment $\mathcal{Z}$ for that purpose and extend the satisfiability relation in [23, Table II] from $(\mathcal{I}, \mathcal{M}, \mathcal{N})$ to $(\mathcal{I}, \mathcal{M}, \mathcal{N}), \mathcal{Z}$. We link satisfiability in the usual way by defining $(\mathcal{I}, \mathcal{M}, \mathcal{N}) \models \varphi$ if $(\mathcal{I}, \mathcal{M}, \mathcal{N}), \mathcal{Z} \models \varphi$ for all $\mathcal{Z}$.

Now, we show that $\mathcal{K}$ can be embedded into an equisatisfiable $\mathcal{SROIQV}(\mathcal{B}^s, \times)\mathcal{K}_{\mathcal{NF}}$ knowledge base $\mathsf{dl}(\mathcal{K})$ over $\Sigma = \langle N_I, N_C, N_R, N_V \rangle$. Again, $N_C = N_{P,1}$ and $N_R = N_{P,2} \cup \{U\} \cup S$, where S is a set of roles defined as in Section 4.1 and $\approx \in N_{P,2}$.

The extension of $\mathsf{dl}(RB)$ to $\mathsf{dl}(\mathcal{K})$ is given below.

1. $\mathsf{dl}(C(t)) := \exists U.(\{t\} \sqcap C)$;
2. $\mathsf{dl}(R(t, u)) := \exists U.(\{t\} \sqcap \exists R.\{u\})$;
3. $\mathsf{dl}(P(t_1, \ldots, t_k)) := \exists U.(\exists P_1.\{t_1\} \sqcap \ldots \sqcap \exists P_k.\{t_k\})$;
4. $\mathsf{dl}(\neg A) := \neg\mathsf{dl}(A)$;
5. $\mathsf{dl}(\mathbf{K}H_1 \vee \mathbf{K}H_l \leftarrow \mathbf{K}A_1, \ldots, \mathbf{K}A_n, \mathbf{not}B_1, \ldots, \mathbf{not}B_m) := \mathbf{K}\mathsf{dl}(A_1) \sqcap \ldots \sqcap \mathbf{K}\mathsf{dl}(A_n) \sqcap \neg\mathbf{A}\mathsf{dl}(B_1) \sqcap \ldots \sqcap \neg\mathbf{A}\mathsf{dl}(B_m) \sqsubseteq \mathbf{K}\mathsf{dl}(H_1) \sqcup \ldots \sqcup \mathbf{K}\mathsf{dl}(H_l)$;
6. $\mathsf{dl}(\mathcal{K}) := \mathcal{O} \cup \{\mathsf{dl}(r) | r \in \mathcal{P}\}$.

The definition of a family fam$(\mathcal{I})$ (Definition 10) is straightforwardly lifted from RB to $\mathcal{K}$. We lift Lemma 2 to sets of interpretations and the expressions appearing in $\mathcal{K}$.

Lemma 3 *Let F in $\mathcal{K}$ be of the form A, $\neg A$, $\mathbf{K}F_1$, or $\mathbf{not}F_1$ where A is an atom, and F_1 of the form A or $\neg A$, $\mathcal{M}$ a set of interpretations of $\mathcal{K}$, $\mathcal{M}_1 = \{\mathcal{J} \mid \mathcal{J} \in fam(\mathcal{I})$ with $\mathcal{I} \in \mathcal{M}\}$, $\mathcal{I} \in \mathcal{M}$, $\mathcal{J} \in fam(\mathcal{I})$, and $\mathcal{Z}$ a variable assignment. The following hold.*

1. $(\mathcal{I}, \mathcal{M}, \mathcal{M}), \mathcal{Z} \models F$ *iff* $\mathsf{dl}(F)^{(\mathcal{J}, \mathcal{M}_1, \mathcal{M}_1), \mathcal{Z}} = \Delta$.
2. $(\mathcal{I}, \mathcal{M}, \mathcal{M}), \mathcal{Z} \not\models F$ *iff* $\mathsf{dl}(F)^{(\mathcal{J}, \mathcal{M}_1, \mathcal{M}_1), \mathcal{Z}} = \emptyset$.

Hybrid MKNF KBs can be embedded into $\mathcal{SROIQV}(\mathcal{B}^s, \times)\mathcal{K}_{\mathcal{NF}}$.

Theorem 4 *Let $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ be a hybrid MKNF KB. $\mathcal{M}$ is an MKNF model of $\mathcal{K}$ if and only $\mathcal{M}_1 = \{\mathcal{J} \mid \mathcal{J} \in fam(\mathcal{I})$ with $\mathcal{I} \in \mathcal{M}\}$ is a hybrid MKNF model of $\mathsf{dl}(\mathcal{K})$.*

Note that in few cases, the embedding does not yield a simple KB (cf. Definition 8). Consider, e.g., $\bot \sqsubseteq \exists U.(\{a\} \sqcap C)$ and $\mathbf{K}D(a) \leftarrow \mathbf{K}C(a)$. This problem remains open for future work.

Examples Returning to the discussion at the end of Section 3, the desired formalization in hybrid MKNF can be obtained as

$$\mathbf{K}C(x) \leftarrow \mathbf{K}\text{HasParent}(x, y), \mathbf{K}\text{HasParent}(x, z), \mathbf{K}(y \not\approx z), \mathbf{not}\text{Married}(y, z).$$

Note again that $C(\text{mary})$ can only be derived if we explicitly know both parents. Using the transformation introduced above, this can be expressed with the desired effect in $\mathcal{SROIQV}(\mathcal{B}^s, \times)\mathcal{K}_{\mathcal{NF}}$.

$$\mathbf{K}\exists U.(\{x\} \sqcap \exists\text{HasParent}.\{y\}) \sqcap \mathbf{K}\exists U.(\{x\} \sqcap \exists\text{HasParent}.\{z\}) \sqcap \mathbf{K}\exists U.(\{y\} \sqcap \exists \not\approx .\{z\}) \sqcap \neg\mathbf{A}\exists U.(\{y\} \sqcap \exists\text{Married}.\{z\}) \sqsubseteq \mathbf{K}\exists U.(\{x\} \sqcap C)$$

To illustrate the coverage of answer set programming, consider the following "classical" example.

$$\mathbf{K}P(a) \leftarrow \mathbf{not}Q(a)$$
$$\mathbf{K}Q(a) \leftarrow \mathbf{not}P(a)$$

This KB has two MKNF models, one in which $P(a)$ is true and $Q(a)$ is false and one which reverses the values, i.e., $\mathcal{M}_1 = \{\mathcal{I} \mid \mathcal{I} \models P(a)\}$ and $\mathcal{M}_2 = \{\mathcal{I} \mid \mathcal{I} \models Q(a)\}$. Sets whose elements do not satisfy either of the two conditions do not satisfy the KB. On the other hand, $\mathcal{M}_3 = \{\mathcal{I} \mid \mathcal{I} \models P(a) \wedge Q(a)\}$ satisfies the KB. Checking minimality, however, reveals that, e.g., $(\mathcal{M}_1, \mathcal{M}_3)$ also satisfies the KB. Since $\mathcal{M}_1$ is a superset of $\mathcal{M}_3$, $\mathcal{M}_3$ is not an MKNF model.

Let us now consider the "intuitive" translation

$$\neg\mathbf{A}(\{a\} \sqcap Q) \sqsubseteq \mathbf{K}(\{a\} \sqcap P)$$
$$\neg\mathbf{A}(\{a\} \sqcap P) \sqsubseteq \mathbf{K}(\{a\} \sqcap Q).$$

The only semantic difference is that in hybrid MKNF truth values $\mathbf{t}$, $\mathbf{u}$, and $\mathbf{f}$ are considered, while MKNF DLs use subsets. E.g., for $\mathcal{M}_1 = \{\mathcal{I} \mid \mathcal{I} \models P(a)\}$, we have that $\mathbf{K}(\{a\} \sqcap P)$ is interpreted as $\{a^{\mathcal{I}}\}$, while $\neg\mathbf{A}(\{a\} \sqcap Q)$ is interpreted as the whole of Δ. This shows immediately that this "intuitive" translation does not work.

Instead we may consider a straightforward extension of the n-nary Datalog translation, which is also used for the proof above of coverage of hybrid MKNF:

$$\neg\mathbf{A}\exists U.(\{a\} \sqcap Q) \sqsubseteq \mathbf{K}\exists U.(\{a\} \sqcap P)$$
$$\neg\mathbf{A}\exists U.(\{a\} \sqcap P) \sqsubseteq \mathbf{K}\exists U.(\{a\} \sqcap Q)$$

With this, we achieve the desired result: for $\mathcal{M}_1 = \{\mathcal{I} \mid \mathcal{I} \models P(a)\}$, $\mathbf{K}\exists U.(\{a\} \sqcap P)$ is interpreted as Δ, because (d, a) is known for each $d \in \Delta$. Likewise, $\neg\mathbf{A}\exists U.(\{a\} \sqcap P)$ is interpreted as $\emptyset$, so both GCIs are satisfied. To ensure minimality of knowledge (and maximality of $\mathcal{M}_1$), consider some strict superset $\mathcal{M}_4$ of $\mathcal{M}_1$. Since $\mathbf{A}$ is interpreted w.r.t. $\mathcal{M}_1$, the interpretation of $\neg\mathbf{A}\exists U.(\{a\} \sqcap Q)$ is Δ. However, $\mathbf{K}\exists U.(\{a\} \sqcap P)$ is interpreted in $\mathcal{M}_4$ as $\emptyset$, since $P(a)$ is no longer known. Hence, $\mathcal{M}_1$ is an MKNF model.

5 Related Work and Conclusions

We have proposed $\mathcal{SROIQV}(\mathcal{B}^s, \times)\mathcal{K}_{\mathcal{NF}}$ as an advance towards a unifying logic for the Semantic Web. It covers a wide variety of languages around OWL, rules, and non-montonicity, which have been discussed in the context of Semantic Web ontology languages. While the coverage of our language depends significantly on the integrative strength of hybrid MKNF, our proposal significantly advances on the latter not only in terms of coverage, but also in that it provides a unified syntax, in the tradition of description logics. This syntax rests crucially on the use of nominal schemas and indeed, as discussed in [20], extending OWL 2 DL with nominal schemas is conceptually and syntactically relatively straightforward.

Most closely related in spirit to our endeavor are probably [4, 5, 23]. However, [4, 5] are less encompassing with respect to the languages which can be embedded in it. We have already discussed in detail hybrid MKNF [23], which is covered by our approach.

So, how might the logic proposed here fare as a unifying logic for the Semantic Web stack? The very concept of such a unifying logic currently acts mainly as a driver for research into ontology language development, and as such provides guidance which ensures that languages do not diverge too widely. In this sense, our proposal is valid. However, at the same time there seems to be little discussion in the community on *requirements* for such a unifying logic. In particular, should the unifying logic merely (or primarily) provide a conceptual underpinning, or should it allow practical ontology modeling?

To serve as a conceptual underpinning, it would be desirable to further extend our proposed language to cover further expressive features discussed in the context of Semantic Web ontology languages. In order to develop it into a practical language, strong reasoning algorithms must be developed and implemented, e.g., by incorporating further ideas from [20] and [14], together with modeling guidance and practical use cases.

We believe that our proposal has potential in either direction.

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Appendix: Omitted Proofs

Lemma 1 *Let $(\mathcal{M}_1, \mathcal{M}_2)$ be a pair of sets of interpretations over Δ such that $\mathcal{M}_1 \supseteq \mathcal{M}_2$ and $(\mathcal{M}_1, \mathcal{M}_2)$ satisfies KB, and let (P, N) be the partition of $MA_\Delta(KB)$ induced by $(\mathcal{M}_1, \mathcal{M}_2)$. Then (P, N) is consistent with KB.*

Proof: The corresponding statement in [7] is Lemma 4.8. The argument is split into Lemmas 4.6 to 4.8 in [7]. Since the proof of this Lemma would be exactly identical to the argument presented in [7], due to our restriction to strictly subjectively quantified KBs, we do not repeat it here, but refer to [7] instead. ■

Theorem 1 *A set $\mathcal{M}$ of interpretations over Δ is an MKNF model for $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KB KB iff the partition of $MA_\Delta(KB)$ induced by $(\mathcal{M}, \mathcal{M})$ satisfies the following conditions:*

(1) (P, N) is consistent with KB;
(2) $\mathcal{M} = \{\mathcal{I} \mid \mathcal{I} \models Ob_K(P, N)\}$;
(3) for each $\mathbf{A}C(x) \in N$, $Ob_K(P, N) \not\models C(x)(P, N)$, and for each $\mathbf{A}R(x, y) \in N$, $Ob_K(P, N) \not\models R(x, y)$;
(4) $Ob_K(P, N) \models Ob_A(P, N)$;
(5) for each partition (P', N') of $MA_\Delta(KB')$, where $KB' = KB \cup \{\mathbf{A}C(x) \mid C(x) \in Ob_K(P, N)\} \cup \{\mathbf{A}R(x, y) \mid R(x, y) \in Ob_K(P, N)\}$, at least one of the following conditions does not hold:
   (a) (P', N') is consistent with KB';
   (b) $Ob_K(P, N) \models Ob_K(P', N')$;
   (c) $Ob_K(P', N') \not\models Ob_K(P, N)$;
   (d) $Ob_K(P, N) \models Ob_A(P', N')$.

Proof: This result corresponds to Theorem 4.9 in [7]. It relies on Lemma 1, and, as pointed out in [7], on two further properties. Namely, if a pair $(\mathcal{M}_1, \mathcal{M}_2)$ induces a partition (P, N) of $MA_\Delta(KB)$, consistent with KB, then $(\mathcal{M}_1, \mathcal{M}_2)$ satisfies KB; furthermore, if $\mathbf{M}_1 = \{\mathcal{I} \mid \mathcal{I} \models Ob_K(P, N)\}$ and $\mathbf{M}_2 = \{\mathcal{I} \mid \mathcal{I} \models Ob_A(P, N)\}$, then $(\mathcal{M}_1, \mathcal{M}_2)$ induces the partition (P, N) of $MA_\Delta(KB)$.

Again, the entire proof argument exactly corresponds to the one in [7], with the only difference that the underlying DL is $\mathcal{SROIQ}(\mathcal{B}^s)$ and not $\mathcal{ALC}$. This does, however, not at all affect the proof itself, so, instead of recalling the argument, we refer to the proof of Theorem 4.9 in [7]. ■

Theorem 2 *Let KB be a simple $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KB. Then the MKNF models $\mathcal{M}$ of KB can be characterized by a finite subset of $MA_\Delta(KB)$.*

Proof: (sketch) Following [7], we can identify certain models with equivalence classes, namely the classes that correspond to models that are equivalent up to renaming, i.e., a bijection for the individuals not occurring in KB is considered. While strictly subjectively quantified $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KBs in general have infinitely many models that are equivalent up to renaming (see [7] for a corresponding example in $\mathcal{ALCK}_{\mathcal{NF}}$), the restriction to simple $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KBs suffices. To see this, note that, when creating candidate models, the individuals explicitly appearing in such a model can be limited to those occurring in KB together with the finitely many ones that can be introduced by the restricted quantifier expressions.

Indeed, the fully spelled out argument of the proof can be obtained by exactly reproducing the tableau algorithm in [7]. This algorithm provides a tableau on modal atoms on top of a "standard" DL tableau reasoner for $\mathcal{ALC}$. Note that the restrictions on the occurrence of modal operators in simple $\mathcal{SROIQ}(\mathcal{B}^s)\mathcal{K}_{\mathcal{NF}}$ KBs are such that they exactly match those in simple $\mathcal{ALC}$ KBs in [7]. Hence, it can be readily applied to our case by just substituting the assumed underlying non-modal $\mathcal{ALC}$ reasoner with one for $\mathcal{SROIQ}(\mathcal{B}^s)$. We refer to [7] for the details instead of recalling them here. ■

Lemma 2 *Let A be an atom in RB, $\mathcal{I}$ an interpretation of RB, $\mathcal{J} \in fam(\mathcal{I})$, and $\mathcal{Z}$ a variable assignment.*

1. $\mathcal{I}, \mathcal{Z} \models A$ if and only if $\mathsf{dl}(A)^{\mathcal{J},\mathcal{Z}} = \Delta$. and
2. $\mathcal{I}, \mathcal{Z} \not\models A$ if and only if $\mathsf{dl}(A)^{\mathcal{J},\mathcal{Z}} = \emptyset$.

Proof: The lemma is the same as that found in [20], though extended here to predicates of arbitrary arity. Since $\mathsf{dl}(A)$ has the form $\exists U.D$, $D^{\mathcal{J},\mathcal{Z}} \neq \emptyset$ implies $\mathsf{dl}(A)^{\mathcal{J},\mathcal{Z}} = \Delta$, and $D^{\mathcal{J},\mathcal{Z}} = \emptyset$ implies $\mathsf{dl}(A)^{\mathcal{J},\mathcal{Z}} = \emptyset$, and so $\mathsf{dl}(A)^{\mathcal{J},\mathcal{Z}} \neq \emptyset$ iff $\mathsf{dl}(A)^{\mathcal{J},\mathcal{Z}} = \Delta$. As noted in [20], we need only show $\mathsf{dl}(A)^{\mathcal{J},\mathcal{Z}} \neq \emptyset$ iff $\mathcal{I}, \mathcal{Z} \models A$. The cases for $A = C(t)$ and $A = R(t, u)$ follow from Lemma 1 in [20] and items (b) an (c) from Definition 10. If $A = P(t_1, \ldots, t_k)$, then we have $(t_1^{\mathcal{I},\mathcal{Z}}, \ldots, t_k^{\mathcal{I},\mathcal{Z}}) \in P^{\mathcal{I}}$ iff there is an $e \in \Delta$ such that $(e, t_i^{\mathcal{J},\mathcal{Z}}) \in P_i^{\mathcal{J},\mathcal{Z}}$, for each i. The claim follows from (d) in Definition 10 and the reduction from $\mathcal{J}$ to $\mathcal{I}$. ■

Theorem 3 *Let RB be a Datalog program. $\mathcal{M}$ is the set of all interpretations $\mathcal{I}$ that satisfy RB if and only if $\mathcal{M}_1 = \{\mathcal{J} \mid \mathcal{J} \in fam(\mathcal{I})$ with $\mathcal{I} \in \mathcal{M}\}$ is the set of all interpretations that satisfy $\mathsf{dl}(RB)$.*

Proof: Consider $\mathcal{I} \in \mathcal{M}$ that satisfies RB, $\mathcal{Z}$ an assignment, and $\mathcal{B} \rightarrow H \in RB$. If $\mathcal{I}, \mathcal{Z} \models H$, then $\mathsf{dl}(H)^{\mathcal{J},\mathcal{Z}} = \Delta$ by Lemma 2. Similarly, if $\mathcal{I}, \mathcal{Z} \not\models \mathcal{B}$, then there exists $A_i \in \mathcal{B}$ such that $\mathcal{I}, \mathcal{Z} \not\models A_i$. Again by Lemma 2, $\mathsf{dl}(A_i)^{\mathcal{J},\mathcal{Z}} = \emptyset$, and so $\mathsf{dl}(\mathcal{B})^{\mathcal{J},\mathcal{Z}} = \emptyset$. Either way, $\mathcal{J}, \mathcal{Z} \models \mathsf{dl}(\mathcal{B} \rightarrow H)$. Generalizing on $\mathcal{Z}$ and $\mathcal{B} \rightarrow H$, $\mathcal{J}$ satisfies $\mathsf{dl}(RB)$.

Now suppose $\mathcal{J} \in \mathcal{M}_1$ satisfies $\mathsf{dl}(RB)$, $\mathcal{B} \rightarrow H \in RB$, and $\mathcal{I}, \mathcal{Z} \models \mathcal{B}$ for some assignment $\mathcal{Z}$. For each $A_i \in \mathcal{B}, \mathcal{I}, \mathcal{Z} \models A_i$. By Lemma 2, $\mathsf{dl}(A_i)^{\mathcal{J},\mathcal{Z}} = \Delta$ for each $A_i \in \mathcal{B}$, and so $\mathsf{dl}(\mathcal{B})^{\mathcal{J},\mathcal{Z}} = \Delta$. Since $\mathcal{J}, \mathcal{Z} \models \mathsf{dl}(\mathcal{B} \rightarrow H)$, $\mathsf{dl}(H)^{\mathcal{J},\mathcal{Z}} = \Delta$, and so $\mathcal{I}, \mathcal{Z} \models H$ by Lemma 2. As such, $\mathcal{I}, \mathcal{Z} \models \mathcal{B} \rightarrow H$. Generalizing on $\mathcal{Z}$ and $\mathcal{B} \rightarrow H, \mathcal{I}$ satisfies RB. ■

Lemma 3 *Let F in $\mathcal{K}$ be of the form A, $\neg A$, $\mathbf{K}F_1$, or $\mathbf{not}F_1$ where A is an atom, and F_1 of the form A or $\neg A$, $\mathcal{M}$ a set of interpretations of $\mathcal{K}$, $\mathcal{M}_1 = \{\mathcal{J} \mid \mathcal{J} \in fam(\mathcal{I})$ with $\mathcal{I} \in \mathcal{M}\}$, $\mathcal{I} \in \mathcal{M}$, $\mathcal{J} \in fam(\mathcal{I})$, and $\mathcal{Z}$ a variable assignment. The following two statements hold.*

1. $(\mathcal{I}, \mathcal{M}, \mathcal{M}), \mathcal{Z} \models F$ iff $\mathsf{dl}(F)^{(\mathcal{J},\mathcal{M}_1,\mathcal{M}_1),\mathcal{Z}} = \Delta$.
2. $(\mathcal{I}, \mathcal{M}, \mathcal{M}), \mathcal{Z} \not\models F$ iff $\mathsf{dl}(F)^{(\mathcal{J},\mathcal{M}_1,\mathcal{M}_1),\mathcal{Z}} = \emptyset$.

Proof: It suffices to show that $\mathsf{dl}(F)^{(\mathcal{J},\mathcal{M}_1,\mathcal{M}_1),\mathcal{Z}} \neq \emptyset$ iff $(\mathcal{I}, \mathcal{M}, \mathcal{M}), \mathcal{Z} \models F$, as in Lemma 2. We consider four cases.

The first case for $F = A$ follows directly from Lemma 2.

In the second case $F = \neg A$, then this amounts to showing that $\mathsf{dl}(A)^{(\mathcal{J},\mathcal{M}_1,\mathcal{M}_1),\mathcal{Z}} = \emptyset$ iff $(\mathcal{I}, \mathcal{M}, \mathcal{M}), \mathcal{Z} \not\models A$ as shown in Tables 1 and 2. We have just shown the contrapositive statement in the first case.

In the third case, $F = \mathbf{K}F_1$, then we have to show that $\bigcap_{\mathcal{J}\in\mathcal{M}_1} \mathsf{dl}(F_1)^{(\mathcal{J},\mathcal{M}_1,\mathcal{M}_1),\mathcal{Z}} \neq \emptyset$ iff $(\mathcal{I},\mathcal{M},\mathcal{M}),\mathcal{Z} \models F_1$ for all $\mathcal{I} \in \mathcal{M}$. Generalizing the proofs on the first two cases, all $\mathcal{I} \in \mathcal{M}$, all $\mathcal{J} \in \mathcal{M}_1$, and the fact that $\mathcal{J} \in \text{fam}(\mathcal{I})$, this case holds as well.

Finally, if $F = \mathbf{not}F_1$, then we have to show that $\Delta \setminus (\bigcap_{\mathcal{J}\in\mathcal{M}_1} \mathsf{dl}(F_1)^{(\mathcal{J},\mathcal{M}_1,\mathcal{M}_1),\mathcal{Z}}) \neq \emptyset$ iff $(\mathcal{I},\mathcal{M},\mathcal{M}),\mathcal{Z} \not\models F_1$ for some $\mathcal{I} \in M$, by Tables 1, 2. The left hand side reduces to $(\bigcap_{\mathcal{J}\in\mathcal{M}_1} \mathsf{dl}(F_1)^{(\mathcal{J},\mathcal{M}_1,\mathcal{M}_1),\mathcal{Z}}) = \emptyset$, proving the claim. ■

Theorem 4 *Let $\mathcal{K} = (\mathcal{O},\mathcal{P})$ be a hybrid MKNF KB. $\mathcal{M}$ is an MKNF model of $\mathcal{K}$ if and only $\mathcal{M}_1 = \{\mathcal{J} \mid \mathcal{J} \in fam(\mathcal{I})$ with $\mathcal{I} \in \mathcal{M}\}$ is a hybrid MKNF model of $\mathsf{dl}(\mathcal{K})$.*

Proof: The proof consists of two parts. In the first part, we show that $\mathcal{M} \models \mathcal{K}$ iff $\mathcal{M}_1 \models \mathsf{dl}(\mathcal{K})$. For that, we consider the two parts $\mathcal{O}$ and $\mathcal{P}$ separately.

If we consider $\mathcal{O}$, then $\mathcal{M} \models \mathbf{K}\pi(\mathcal{O})$ iff $\mathcal{M}_1 \models \mathcal{O}$ by the definition on page 7 and the definition of $\mathsf{dl}(\mathcal{K})$. Since $\mathcal{O}$ is closed, $\mathcal{M} \models \mathbf{K}\pi(\mathcal{O})$ iff $\mathcal{M} \models \pi(\mathcal{O})$. Now, the claim follows straightforwardly from the first-order semantics and the definition of fam$(\mathcal{I})$ for each $\mathcal{I} \in \mathcal{M}$, since $\mathcal{O}$ is free of modal operators and predicates of arity greater 2.

Next, consider the set of MKNF rules $\mathcal{P}$. The claim can be directly obtained from the corresponding statement for Datalog rules (Theorem 3) and the fact that first-order atoms in Datalog and modal atoms in MKNF rules satisfy exactly the same necessary property (Lemma 3 as extension of Lemma 2).

In the second part, we show that, indeed, $\mathcal{M}$ is an MKNF model of $\mathcal{K}$ iff $\mathcal{M}_1$ is an MKNF model of $\mathsf{dl}(\mathcal{K})$ considering condition (2) in Definition 5 and the definition on page 7. Both directions have to be considered.

Suppose that $\mathcal{M}$ is an MKNF model of $\mathcal{K}$, i.e., $\mathcal{M}$ satisfies $\mathcal{K}$, and for each set of interpretations $\mathcal{M}'$ with $\mathcal{M}' \supset \mathcal{M}$ we have $(\mathcal{I}',\mathcal{M}',\mathcal{M}) \not\models \varphi$ for some $\mathcal{I}' \in \mathcal{M}'$. We already know from the first part of the proof that $\mathcal{M}_1 \models \mathsf{dl}(\mathcal{K})$.

Assume that $\mathcal{M}_1$ is not an MKNF model of $\mathsf{dl}(\mathcal{K})$. There is $\mathcal{M}_1'$ with $\mathcal{M}_1 \subset \mathcal{M}_1'$, and $(\mathcal{J}',\mathcal{M}_1',\mathcal{M}_1) \models \mathsf{dl}(\mathcal{K})$ for all $\mathcal{J}' \in \mathcal{M}_1'$ by Definition 5. We can construct a set $\mathcal{M}'$ by reducing all $\mathcal{J}' \in \mathcal{M}_1'$ to interpretations $\mathcal{I}'$ for $\mathcal{K}$ with $\mathcal{I}' \in \mathcal{M}'$. It remains to show that $(\mathcal{I}',\mathcal{M}',\mathcal{M}) \models \mathcal{K}$ for all $\mathcal{I}' \in \mathcal{M}'$ and that $\mathcal{M} \subset \mathcal{M}'$. Then, by the definition on page 7, we derive a contradiction to $\mathcal{M}$ being an MKNF model of $\mathcal{K}$.

We show first, that $(\mathcal{I}',\mathcal{M}',\mathcal{M}) \models \mathcal{K}$ for all $\mathcal{I}' \in \mathcal{M}'$. Note that $\mathcal{O}$ itself is free of modal operators (and closed), so $(\mathcal{I}',\mathcal{M}',\mathcal{M}) \models \mathcal{O}$ can be derived from $(\mathcal{J}',\mathcal{M}_1',\mathcal{M}_1) \models \mathsf{dl}(\mathcal{K})$, given the construction of $\mathcal{M}'$, and similar to the case of $\mathcal{O}$ in the first part of the proof. Thus, $(\mathcal{I}',\mathcal{M}',\mathcal{M}) \models \mathcal{O}$ holds for all $\mathcal{I}' \in \mathcal{M}'$. Moreover, Lemma 3, can be adapted to this case, e.g., the condition $((\mathcal{I}',\mathcal{M}',\mathcal{M}), Z \models F$ iff $\mathsf{dl}(F)^{(\mathcal{J}',\mathcal{M}_1',\mathcal{M}_1),\mathcal{Z}} = \Delta)$ can be shown, and likewise the other in Lemma 3 for structures $(\mathcal{I}',\mathcal{M}',\mathcal{M})$, $(\mathcal{J}',\mathcal{M}_1',\mathcal{M}_1)$. Thus, $(\mathcal{I}',\mathcal{M}',\mathcal{M}) \models r$ holds for each rule $r \in \mathcal{P}$ since $(\mathcal{J}',\mathcal{M}_1',\mathcal{M}_1) \models \mathsf{dl}(r)$ for each $r \in \mathcal{P}$, following the same argument as in the first part of the proof dealing with a set of MKNF rules $\mathcal{P}$. This shows that $(\mathcal{I}',\mathcal{M}',\mathcal{M}) \models \mathcal{P}$ for all $\mathcal{I}' \in \mathcal{M}'$, which proves the claim.

Finally, we show that $\mathcal{M} \subset \mathcal{M}'$. We have $\mathcal{M}_1 = \{\mathcal{J} \mid \mathcal{J} \in \text{fam}(\mathcal{I})$ with $\mathcal{I} \in \mathcal{M}\}$. Since fam$(\mathcal{I})$ is defined to contain all possible $\mathcal{J}$ for all $\mathcal{I} \in \mathcal{M}$, there exists $\mathcal{J}'$ with $\mathcal{J}' \in \mathcal{M}_1' \setminus \mathcal{M}_1$ such that $\mathcal{J}' \notin \text{fam}(\mathcal{I})$ for any $\mathcal{I} \in \mathcal{M}$. Since $\mathcal{M}_1$ and $\mathcal{M}_1'$ can be reduced to $\mathcal{M}$ and $\mathcal{M}'$, we obtain $\mathcal{M}' \supset \mathcal{M}$,

The other direction follows from a similar argument. ■