Radial Restraint: 
A Semantically Clean Approach to Bounded Rationality for Logic Programs

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Abstract
Declarative logic programs (LP) based on the well-founded semantics (WFS) are used for knowledge representation (KR), e.g., in databases, business rules, semantic web, and the SILK system. They represent logical non-monotonicity, and offer much better scalability than answer-set programs or first-order logic. In this paper, we present radial restraint: a novel approach to bounded rationality in LP. Radial restraint is parameterized by a norm that measures the syntactic complexity of a term, along with an abstraction function based on that norm. When a term exceeds a bound for the norm, the term is assigned the WFS’s third truth-value of undefined. If the norm is finitary, radial restraint guarantees finiteness of models and decidability of inferencing, even when logical functions are present. It further guarantees soundness, even when non-monotonicity is present. We give a fixed-point semantics for radially restrained well-founded models which soundly approximate well-founded models. We also show how to perform correct inferencing relative to such models, via SLGABS, an extension of classical SLG resolution that uses norm-based abstraction functions. Finally we discuss how SLGABS is implemented in the engine of XSB Prolog, and scales to knowledge bases with more than $10^3$ rules and facts.

Introduction
While AI bounded rationality research has largely focused on decision-theoretic optimization (e.g., (Russell and Subramanian 1995; Anderson and Oates 2007)), a strand has focused on limiting reasoning via deduction principles that derive some beliefs explicitly but leave others implicit (Konev 1983; Levesque 1984; Fisher and Ghidini 1999; Grant, Kraus, and Perlis 2000; Fisher et al. 2007). To date, however, this strand has lacked much practical impact. For resource-limiting logic programming (LP), the main approach that has emerged in practice is to set (manually or heuristically) a inferencing engine parameter — for instance, a timeout or a term-depth bound in Prolog — and to treat as false any atom that is not inferred before the parameter bound is exceeded. However, incompleteness about an atom $A$ can lead to unsoundness if another atom $A'$ depends negatively on $A$. In addition, the results of such inferencing depend on the implementation code or session. Radial restraint overcomes both these shortcomings. It introduces (to our knowledge) the use of the truth value undefined to represent implicit deductions that have not been made explicit.

Our starting point is recent work on termination properties of logic programs with negation. When tabled evaluation is extended with subgoal abstraction, first introduced in (Tamaki and Sato 1986), tabling can ensure termination to queries to safe normal programs that are strongly bounded term size (SBTS) (Riguzzi and Swift 2013b; 2013a). Such programs have well-founded models (van Gelder, Ross, and Schlipf 1991) that are representable via finite sets of true and undefined ground atoms, and have been shown to properly include the finitely ground programs of (Calimeri et al. 2008), a class motivated by the needs of ASP grounders (cf. (Alviano, Faber, and Leone 2010)). Although these results are powerful, they do have drawbacks. It is not decidable whether a program is SBTS. In addition, while SBTS-programs are Turing-complete (shown for finitely ground programs in (Calimeri et al. 2008)), some natural programs are not SBTS: such as those that contain a predicate to determine list membership. From a theoretical level, this weakness can be addressed by defining program classes that terminate for various types of queries (cf. e.g., (Bonatti 2004)). However, membership in such classes is again not decidable.

The tabled evaluation method of (Riguzzi and Swift 2013a) is complete for SBTS-programs and queries. Building on this evaluation method, and making use of the undefined truth value as discussed above, we ensure that both the evaluations and the (sub-)models they produce are finite.

We introduce aspects of the approach through the following example, to motivate the formalism that follows.

Example 1 Consider the program $P_{inf}$:

\[
\begin{align*}
\text{p(s(X))} & \leftarrow \text{p(X)}. \\
\text{p(0)}. \\
\text{q(0)}. 
\end{align*}
\]

$P_{inf}$ is not SBTS as the sets both of true atoms of its well-founded model, $true(WFM^{inf})$, and the false atoms, $false(WFM^{inf})$, are infinite. However, an approximation of the answers to $p(X)$ can be made by restraining inferencing. For instance, if a depth norm were used with a level of 4, then the answers $p(0)$, $p(s(0))$, and $p(s(s(0)))$ would be derived. However the answer $p(s(s(s(0))))$ would be abstracted to $p(s(s(s(X))))$ and all ground atoms unifying with this answer would be assigned the truth value of undefined.
This infinite set of answers is represented by the sentence \( \forall X. p(s(s(s(X)))) \). By allowing the set of atoms in WFM\(^{P_{rad}}\) whose truth value is undefined (undef\( (\text{WFM}\, P_{rad}) \)) to be represented by such sentences in addition to atoms, a finite representation of the radially restrained model is constructed. In addition, because the default negation of any undefined atom is itself undefined in the well-founded semantics, the answer abstraction preserves the soundness of negation. For instance the literal not \( p(s(s(s(0)))) \) is also assigned undefined.

Intuitively, atoms that are true or false in the well-founded model of a program remain true or false in the radially restrained model as long as they don’t exceed a bound defined by a norm on atoms. Those atoms that exceed the bound are abstracted, and have the truth value undefined. In this way, radially restrained models are sound approximations to the well-founded model. Further, because variables in the abstraction are regarded as universally quantified, the resulting model can be represented using finite sets of true and undefined atoms.

This paper thus explores radially restrained models along with their efficient query evaluation. Specifically,

- We define the radially restrained well-founded model of a normal program \( P \) as parameterized by an abstraction function \( \text{abs}(\cdot) \). We show that such a model soundly approximates the well-founded model of \( P \), and that if \( \text{abs}(\cdot) \) is replaced by a weaker abstraction, the approximation of the radially restrained model becomes tighter.
- By extending SLG resolution with subgoal abstraction (Riguzzi and Swift 2013a) to incorporate an abstraction function for answers, we introduce SLG\(_{ABS} \), which correctly evaluates queries with respect to radially restrained models. Given a finitary abstraction function, SLG\(_{ABS} \) terminates with an asymptotic complexity that is equal to the best complexity that is known.

Finally, based on the the SLG-WAM of XSB (Swift and Warren 2012), we describe an implementation of SLG\(_{ABS} \) that is declarative, efficient and scalable.

**Background**

We assume a general knowledge of logic programming terminology, including tabled resolution and the well-founded semantics. In addition we make use of the following terminology, including tabled resolution and the well-founded semantics.

Throughout this paper we restrict our attention to normal programs, and to queries and subgoals that are atoms. We also assume a fixed strategy for selecting literals in a clause: without loss of generality we assume the selection strategy is left-to-right. In accordance with this strategy, a normal rule has the form

\[ r = A_0 \leftarrow A_1, \ldots, A_m, \text{not } A_{m+1}, \ldots, \text{not } A_n \]

where \( A_0, \ldots, A_n \) are atoms. A program \( P \) is safe if each rule \( r \) in \( P \) is such that every variable in \( r \) occurs in a positive literal in the body of \( r \). Our attention is also restricted to three-valued (partial) interpretations and models, such as the well-founded model. Each such interpretation is represented as a pair of true and false atoms: \( \langle \text{true}(I); \text{false}(I) \rangle \). For two interpretations, \( I \) and \( J \), \( I \subseteq J \) iff \( \text{true}(I) \subseteq \text{true}(J) \) and \( \text{false}(I) \subseteq \text{false}(J) \). Alternatively, a three-value interpretation can be represented as a set of literals.

Symbols within a term may be represented through positions which are members of the set \( \Pi \). A position in a term is either the empty string \( \Lambda \) that reaches the root of the term, or the string \( \pi.i \) that reaches the \( i \)th child of the term reached by \( \pi \), where \( \pi \) is a position and \( i \) an integer. For a term \( t \) we denote the symbol at position \( \pi \) in \( t \) by \( t_\pi \). For example, \( p(a, f(X))_2.1 = X \). We assume that a program \( P \) is defined over a language \( L \), containing a finite set \( P_N \) of predicate and function symbols, and a countable set of variables from the set \( V \cup \hat{V} \). Elements of the set \( V \) are referred to as program variables. Elements of the set \( \hat{V} \), called position variables, are of the form \( X_\pi \), where \( \pi \) is a position. These variables are used when it is convenient to mark certain positions of interest in a term. The Herbrand Universe of \( L \) is denoted \( H_L \), or as \( H_P \) if \( L \) consists of the predicate and function symbols in \( P \); similarly the Herbrand Base is denoted as \( B_L \) or as \( B_P \). Throughout the paper variant terms are considered to be equal.

**Dynamic Stratification**

One of the most important formulations of stratification is that of dynamic stratification. (Przymusinski 1989) shows that a program has a 2-valued well-founded model iff it is dynamically stratified, so that it is the weakest notion of stratification that is consistent with the well-founded semantics. As presented in (Przymusinski 1989), dynamic stratification computes strata via operators on interpretations of the form \( \langle Tr; Fa \rangle \), where \( Tr \) and \( Fa \) are subsets of \( H_P \).

**Definition 1**

For a normal program \( P \), sets \( Tr \) and \( Fa \) of ground atoms and a 3-valued interpretation \( I \) (sometimes called a pre-interpretation):

\[
\text{True}^P_I(Tr) = \{ A | A \text{ is not true in } I \}; \text{ and there is a clause } B \leftarrow L_1, \ldots, L_n \text{ in } P \text{, a grounding substitution } \theta \text{ such that } A = B\theta \text{ and for every } 1 \leq i \leq n \text{ either } L_i\theta \text{ is true in } I, \text{ or } L_i\theta \in Tr \};
\]

\[
\text{False}^P_I(Fa) = \{ A | A \text{ is not false in } I \}; \text{ and for every clause } B \leftarrow L_1, \ldots, L_n \text{ in } P \text{ and grounding substitution } \theta \text{ such that } A = B\theta \text{ there is some } i (1 \leq i \leq n) \text{ such that } L_i\theta \text{ is false in } I \text{ or } L_i\theta \in Fa \}.
\]

(Przymusinski 1989) shows that \( \text{True}^P_I \) and \( \text{False}^P_I \) are both monotonic, and defines \( T^P_I \) as the least fixed point of \( \text{True}^P_I(\theta) \) and \( F^P_A \) as the greatest fixed point of \( \text{False}^P_I(H_P) \). In words, the operator \( T^P_I \) extends the interpretation \( I \) to add the new atomic facts that can be derived from \( P \) knowing \( I \); \( F^P_A \) adds the new negations of atomic facts that can be shown false in \( P \) by knowing \( I \) (via the uncovering of unfounded sets). An iterated fixed point operator builds up dynamic strata by constructing successive partial interpretations as follows.

**Definition 2 (Iterated Fixed Point and Dynamic Strata)**

For a normal program \( P \) let
Norms and Abstractions

Abstraction functions may be understood with respect to norms, which can specify families of abstraction functions. Typically, if the norm of an atom $A$ is greater than a given integer bound, $A$ is abstracted.

A norm $N(\cdot)$ is a function from terms to non-negative integers such that

1. $N(t) = 0$ iff $t = \Lambda$ (the empty term)
2. $t$ subsumes $t'$ implies $N(t) \leq N(t')$

A norm is finitary iff for any finite non-negative integer $k$, the cardinality of the set $\{t | t \in H_L \land N(t) < k\}$ is finite.

An abstraction of a term $t$, denoted $\text{abs}(t)$, may replace subterms of $t$ by position variables: formally, $\text{abs}(t)$ is a term such that if $\text{abs}(t)|_{\pi} \in (FN \cup V)$, then $\text{abs}(t)|_{\pi} = t|_{\pi}$. For instance $p(f(g(X_{1,1,1}), X_{1,2}), X_2)$ is an abstraction of $p(f(g(a), X), X)$. It is easy to see that $\text{abs}(t)$ subsumes $t$, so for any norm $N(\cdot)$, $N(\text{abs}(t)) \leq N(t)$. An abstraction $\text{abs}(\cdot)$ is finitary if the cardinality of $\{\text{abs}(t) | t \in H_L\}$ is finite. Given two abstractions, $\text{abs}_1(\cdot) \leq \text{abs}_2(\cdot)$ if for all terms $t$, $\text{abs}_1(t)$ subsumes $\text{abs}_2(t)$. Note that if $\text{abs}_1(\cdot) \leq \text{abs}_2(\cdot)$, then $\{\text{abs}(t) | t \in H_L\} \subseteq \{\text{abs}(t) | t \in H_L\}$. Norms and abstractions are applied to atoms by taking those atoms as terms, and to rules by applying the operation to each atom underling a literal in the rule.

Example 2 A depth norm, $\text{depth}(\cdot)$, maps a term $t$ to the maximal depth of any position in $t$, where the depth of the outermost function symbol of $t$ is 1 and the depth of a position $\pi,i$ is the depth of $\pi$ plus 1 if $t|_{\pi,i}$ is a not a position variable, and is the depth of $\pi$ otherwise. For a positive integer $k$, a depth-$k$ abstraction is an abstraction that maps $t$ to itself if $\text{depth}(t)$ is less than or equal to $k$; and otherwise to the abstraction of $t$ with depth $k$ that is maximal with respect to subsumption. It is easy to see that such a maximal depth-$k$ abstraction of $t$ must be unique. Within the atom $A = p(a, f(b, g(c)))$ the depth of $c$ is 4. The depth 3 abstraction of $A$ is $p(a, f(b, g(X_{2,2,1})))$, and the depth 2 abstraction of $A$ is $p(a, f(X_{2,1}, X_{2,2}))$. Both the depth norm and the family of depth-$k$ abstractions (for positive integer $k$) are finitary.

Depth-$k$ abstractions are simple to understand and to implement. However the number of terms whose depth is less than $k$ may grow exponentially. Thus, other abstractions, based on the size of a term, or that weigh the occurrence of certain types of function symbols over others (e.g., lists) can be practically useful. Finally, note that the identity function on terms, $I(\cdot)$, is an abstraction function, but is not finitary for languages that contain non-constant function symbols. In fact, $I(\cdot)$ is the maximal abstraction function.

Radially Restrained Models

The operators for Radially Restrained Models are based on abstraction functions rather than on norms, both to provide a basis for Theorem 2, and to highlight the correspondence with the resolution method described in the next section.

Definition 3 For a normal program $P$, abstraction function $\text{abs}(\cdot)$, sets $\text{Tr}$ and $\text{Fa}$ of ground atoms, and a 3-valued interpretation $I$ (sometimes called a pre-interpretation):

$\text{True}_P^I(\text{abs}(B), \text{Tr}) = \{A | \text{there is a clause } B \leftarrow L_1, \ldots, L_n \text{ in } P, \text{ a grounding substitution } \theta \text{ such that } A = B\theta \land \text{abs}(B\theta) \in \text{Tr} \land \text{for every } 1 \leq i \leq n \text{ either } L_i\theta \text{ is true in } I \text{ or } L_i\theta \notin \text{Tr}\}$;

$\text{False}_P^I(\text{abs}(B), \text{Fa}) = \{A | \text{for every clause } B \leftarrow L_1, \ldots, L_n \text{ in } P \text{ and a grounding substitution } \theta \text{ such that } A = B\theta \land \text{abs}(B\theta) \text{ and there is some } i \leq \alpha \text{ such that } L_i\theta \text{ is false in } I \text{ or } L_i\theta \in \text{Fa}\}$.

Unlike Definition 1, Definition 3 requires that $\text{abs}(B\theta) = B\theta$ in order for an atom to be considered either true or false. Clearly both $\text{True}_P^I$ and $\text{False}_P^I$ are monotonic in their second arguments; and as with the well-founded model, we define $\text{Tr}_P^I(\text{abs})$ as the least fixed point of $\text{True}_P^I(\text{abs}, \emptyset)$ and $\text{Fa}_P^I(\text{abs})$ as the greatest fixed point of $\text{False}_P^I(\text{abs}, \emptyset)$.

Definition 4 (Radially Restrained Model) For a normal program $P$ and abstraction function $\text{abs}(\cdot)$

$\text{WFM}_0 = \langle \emptyset; \emptyset \rangle$;

$\text{WFM}_{n+1} = \text{WFM}_n \cup \{\text{Tr}_P^{\text{abs}(\cdot)}(\text{abs}); \text{Fa}_P^{\text{abs}(\cdot)}(\text{abs})\}$;

$\text{WFM}_n = \bigcup_{\beta < \alpha} \text{WFM}_\beta$, for limit ordinal $\alpha$.

The radially restrained model $\text{WFM}(\text{abs}, P)$ denotes the fixed point interpretation $\text{WFM}_\delta$, where $\delta$ is the smallest (countable) ordinal such that both sets $\text{Tr}_P^{\text{abs}(\cdot)}(\text{abs})$ and $\text{Fa}_P^{\text{abs}(\cdot)}(\text{abs})$ are empty.

The following statement follows directly from Definition 3. Since the language of $P$ has a finite number of function symbols and predicates, and since $\text{abs}(\cdot)$ is finitary, $\text{True}_P^I(\text{abs}, \text{Tr})$ can only produce a finite number of grounded rules, even if $I$ or $\text{Tr}$ were infinite 1.

Proposition 1 For a program $P$ and finitary abstraction function $\text{abs}(\cdot)$ let

$\text{WFM}(\text{abs}, P) = \langle \text{TrueAtoms}; \text{FalseAtoms} \rangle$.

The cardinality of $\text{TrueAtoms}$ is finite.

Footnote 1: Proofs of all results, along with a full presentation of SLG$_{ABS}$ (introduced in the next section) are available at http://www.cs.sunysb.edu/~tswift/webpapers/radial.pdf.
Because \( TR(\text{abs}) \) is monotonic, due to Proposition 1 it must reach fixed point for some finite ordinal. Accordingly, if \( \text{abs}(\cdot) \) is finitary, \( WFM(\text{abs}, P) \) will also reach fixed point at some finite ordinal.

**Theorem 1** Given a program \( P \) and finitary abstraction function \( \text{abs}(\cdot) \), then \( WFM(\text{abs}, P) = WFM(\text{abs}, P)_\delta \) for some finite ordinal \( \delta \).

The main theorem about radially restrained well-founded models is as follows.

**Theorem 2** Let \( \text{abs}_1(\cdot), \text{abs}_2(\cdot) \) be abstraction functions such that \( \text{abs}_1(\cdot) \leq \text{abs}_2(\cdot) \). Then for any program \( P \), \( WFM(\text{abs}_1, P) \subseteq WFM(\text{abs}_2, P) \).

Since the identity function, \( \text{Id}(\cdot) \) is the maximal abstraction function, and since \( WFM(\text{Id}, P) = WFM(P) \), Theorem 2 implies:

**Corollary 1** For a program \( P \) and abstraction function \( \text{abs}(\cdot) \), \( WFM(\text{abs}, P) \subseteq WFM(P) \).

For any program \( P \), Theorem 2 also implies that a chain of abstraction functions \( \text{abs}_1(\cdot), \text{abs}_2(\cdot), \ldots \) such that for \( i \leq j \), \( \text{abs}_i(\cdot) \leq \text{abs}_j(\cdot) \) is associated with a chain of models: \( WFM(\text{abs}_1, P), WFM(\text{abs}_2, P), \ldots, WFM(\text{abs}_j, P) \) such that for \( i \leq j \), \( WFM(\text{abs}_i, P) \subseteq WFM(\text{abs}_j, P) \). Thus, families of finitary abstraction functions, based on depth, size or other measures, provide successively more powerful finite approximations of the well-founded model.

### Tabled Resolution for Bounded Rationality

\( \text{SLG}_{\text{ABS}} \) is a tabled resolution method that correctly evaluates queries to radially restrained models of programs. \( \text{SLG}_{\text{ABS}} \) strictly extends SLG evaluation (Chen and Warren 1996) which models well-founded computation at an operational level, ensuring goal-directedness, termination and optimal complexity for a normal programs. SLG evaluation, along with numerous extensions of it, are well-described in the literature. Accordingly in this section we present only those extensions used in \( \text{SLG}_{\text{ABS}} \), after a brief review of the terminology required by the extensions.

### Terminology Used

In the forest-of-trees model of SLG (Swift 1999), an evaluation is a possibly transfinite sequence of forests (sets) of trees in which each tree corresponds to a subgoal that has been encountered in an evaluation. When a new tabled subgoal \( S \) is encountered, a tree with root \( S \) is added to the current forest by a new \textsc{Subgoal} operation, and children of the root are added through \textsc{Program Clause Resolution} operations. Other positive selected literals are resolved through the \textsc{Positive Return} operation; while ground negative selected subgoals are resolved through the \textsc{Negative Return} operation, or their resolution may be delayed through the \textsc{Delaying} operation. These delayed literals may later be evaluated through \textsc{Simplification} or \textsc{Answer Completion} operations. The need to delay some literals arises because modern Prolog engines rely on a fixed order for selecting literals in a rule. However, well-founded computations cannot be performed using a fixed-order literal selection function. When it is determined that no more resolution may be performed for non-delayed literals in nodes of trees for a mutually dependent set of subgoals, the trees are marked as complete using the \textsc{Completion} operation. If a subgoal \( S \) has been marked as complete and \( S \) has no answers, literals of the form \textit{not} \( S \) can be resolved away by the \textsc{Negative Return} operation.

More specifically, the nodes in each tree have the form

\[
\text{Ans} \leftarrow \text{Delays}\text{Goals} \text{ or } \text{fail}.
\]

In the first form, \( \text{Ans} \) is an atom while \( \text{Delays} \) and \( \text{Goals} \) are sequences of literals. The second form is called a failure node. \( \text{Goals} \) represents the sequence of literals left to be examined, while \( \text{Delays} \) represents those literals that have been examined, but their resolution delayed. A node \( N \) is an answer when it is a leaf node for which \( \text{Goals} \) is empty. If the \( \text{Delays} \) of an answer is empty, it is termed an unconditional answer, otherwise, it is a conditional answer.

\( \text{SLG resolution} \) is used to resolve an answer \( A \) against a node \( N \).

**Definition 5 (SLG Resolution)** Let \( N \) be a node \( A \leftarrow D(L_1, \ldots, L_n) \), where \( n > 0 \). Let \( \text{Ans} = A' \leftarrow D' \) be an answer whose variables are disjoint from \( N \). If \( \exists i, 1 \leq i \leq n \) such that \( L_i \) and \( A' \) are unifiable with mgu \( \theta \), then the resolvent of \( N \) and \( \text{Ans} \) on \( L_i \) has the form:

\[
(A \leftarrow D|L_1, \ldots, L_{i-1}, L_{i+1}, \ldots, L_n)\theta
\]

if \( D' \) is empty; otherwise the resolvent has the form:

\[
(A \leftarrow D, L_i|L_1, \ldots, L_{i-1}, L_{i+1}, \ldots, L_n)\theta.
\]

Example 3 below further illustrates the foregoing concepts within an \( \text{SLG}_{\text{ABS}} \) evaluation.

**Definition 6** Let \( F \) be an SLG forest. The interpretation induced by \( F, \mathcal{I}_F \), is the smallest set such that:

- A (ground) atom \( A \in \text{true}(\mathcal{I}_F) \) iff \( A \) is in the ground instantiation of an unconditional answer \( \text{Ans} \leftarrow \text{in} \in F \).
- A (ground) atom \( A \in \text{false}(\mathcal{I}_F) \) iff \( A \) is in the ground instantiation of a subgoal whose tree in \( F \) is marked as complete, and \( A \) is not in the ground instantiation of any answer in a tree in \( F \).

An atom \( S \) is successful (resp. failed) in \( F \) if \( S' \) is in \( \text{true}(\mathcal{I}_F) \) (\text{false}(\mathcal{I}_F)) for every \( S' \) in the ground instantiation of \( S \). A non-ground subgoal not \( S \) succeeds (fails) if \( S \) fails (succeeds). Given an interpretation \( \mathcal{J} \) and forest \( F \), the restriction of \( \mathcal{J} \) to \( F \), \( \mathcal{J}_F \) is the interpretation such that \( \text{true}(\mathcal{J}_F) \) (\text{false}(\mathcal{J}_F)) consists of those atoms in \( \text{true}(\mathcal{J}) \) (\text{false}(\mathcal{J})) that are in the ground instantiation of some subgoal whose tree is in \( F \).

\( \text{SLG}_{\text{ABS}} \)

\( \text{SLG}_{\text{ABS}} \) extends SLG to use abstraction both when creating a tree for a new subgoal, and when deriving an answer.
Definition 7 (Subgoal Abstraction (Riguzzi and Swift 2013a))

NEW SUBGOAL: Let \( \text{abs}(\cdot) \) be an abstraction function, and let a forest \( \mathcal{F}_n \) contain a tree with non-root node

\[
N = \text{Ans} \leftarrow \text{Delays}[G, \text{Goals}]
\]

where \( S \) is the underlying subgoal of the literal \( G \). Assume \( \mathcal{F}_n \) contains no tree with root \( \text{abs}(S) \). Then add the tree \( \text{abs}(S) \leftarrow |\text{abs}(S)| \) to \( \mathcal{F}_n \).

Abstraction is also used when an answer \( \text{Ans} \) is derived; if the abstraction is non-trivial, i.e., if \( \text{Ans} \neq \text{abs}(\text{Ans}) \), then a special atom \( \text{undefined}_\text{abs} \) is added to the Delays of \( \text{Ans} \).

Definition 8 (Answer Abstraction)

POSITIVE RETURN: Let \( \text{abs}(\cdot) \) be an abstraction function, and let a forest \( \mathcal{F}_n \) contain a tree with non-root node \( N \) whose selected literal \( S \) is positive. Let \( \text{Ans} \) be an answer for \( S \) in \( \mathcal{F}_n \) and \( N' = A \leftarrow \text{Delays}[\text{Goals}] \) be the SLG resolvent of \( N \) and \( \text{Ans} \) on \( S \).

- If \( \text{Goals} \) is non-empty, then \( N'_{\text{child}} = N' \);
- Otherwise, if \( \text{abs}(N') = N' \), then \( N'_{\text{child}} = \text{abs}(A) \leftarrow \text{Delays}, \text{undefined}_\text{abs} \);
- Otherwise, if \( \text{abs}(N') \neq N' \), \( N'_{\text{child}} = \text{abs}(A) \leftarrow \text{Delays}, \text{undefined}_\text{abs} \).

If \( N \) does not have a child \( N'_{\text{child}} \) in \( \mathcal{F}_n \), then add \( N'_{\text{child}} \) as a child of \( N \).

For SLG\textsubscript{ABS} to be correct with respect to radically restrained models of normal programs, negation must be extended to handle the lack of safety that is introduced by abstraction. The following example shows how this can occur, and illustrates the SLG and SLG\textsubscript{ABS} terminology used so far.

Example 3 Figure 1 shows the SLG\textsubscript{ABS} evaluation of the query \( r(X) \) against the safe program \( P_{\text{abs}-\text{unsafe}} \).

\[
p(s(X)) \leftarrow p(X),
p(0).
\]

\[
r(X) \leftarrow p(X), \neg q(X),
q(0).
\]

where a depth-2 abstraction function is used (local scheduling is assumed for this evaluation, cf. (Swift and Warren 2012)). The evaluation begins in a manner identical to SLG evaluation. The initial forest consists simply of node 0. Children of root nodes are created by \textsc{Program Clause Resolution}, which creates node 1. The selected (leftmost) literal of node 1 is \( p(X) \), which is new at this point of the evaluation. A \textsc{New Subgoal} operation creates node 2, (although an abstraction is applied, it is trivial), and \textsc{Program Clause Resolution} creates node 3, an unconditional answer. Reapplication of \textsc{Program Clause Resolution} also creates node 4, whose selected literal is not new to the evaluation. There is already an answer for \( p(X) \) so that \textsc{Positive Return} is applicable to this node; repeated applications of \textsc{Positive Return} produce nodes 5 and 6. Although abstraction is performed for all answers, it is trivial except when producing node 6. Once node 6 is produced, the tree for \( p(X) \) is completely evaluated, and a \textsc{Completion} operation marks it complete. Another \textsc{Positive Return} operation produces node 7 which has a selected negative literal. Evaluation of the subgoal \( q(0) \) shows that \( q(0) \) is successful, and a \textsc{Negative Return} operation creates a failure node as child 10. The evaluation proceeds until finally the conditional answer; node 6 is resolved against the selected literal of node 14. Because the answer was conditional, the selected literal \( p(s(s(X)) \) is moved to the Delays after resolution (Definition 5). Because of the abstraction used to produce node 6, the next selected literal not \( q(s(s(X)) \) is non-ground. Nonetheless, the atom \( q(s(s(X)) \) becomes failed (Definition 6), once its tree is completed with no answers (step 15a). Because \( q(s(s(X)) \) is failed, a \textsc{Negative Return} operation resolves the selected literal away, leading to the conditional answer node 16.

Thus SLG\textsubscript{ABS} has the following extensions over SLG:

1. Abstraction is used both when creating new trees (in the \textsc{New Subgoal} operation), and when adding an answer (in the \textsc{Positive Return} operation);
2. A special atom \( \text{undefined}_\text{abs} \) is added to the Delays of each non-trivially abstracted answer \( A \) (in the \textsc{Positive Return} operation). The truth value of \( \text{undefined}_\text{abs} \) is always \( \text{undefined} \), so it can never be removed from the Delays of \( A \), forcing \( A \) to have a truth value of \( \text{undefined} \) as well; and
3. \textsc{Negative Return} is defined so that literal \( \text{not} \ A \) can be resolved away in a forest \( \mathcal{F} \) if \( A \) is failed in \( \mathcal{F} \), regardless of whether \( A \) is ground.

Of course, \textsc{New Subgoal} and \textsc{Positive Return} in SLG\textsubscript{ABS} can be reduced to the classical definitions of SLG by setting the \( \text{abs}() \) to the identity function.

If a finitary abstraction function is used in SLG\textsubscript{ABS}, then any forest has a finite number of trees and answers. This fact together with other tabling properties ensures the following.

Theorem 3 Let \( Q \) be a query to a normal program \( P \), and let \( \text{abs}(\cdot) \) be a finitary abstraction function. Then any SLG\textsubscript{ABS} evaluation \( \mathcal{E} \) of \( Q \) against \( P \) reaches a final forest \( \mathcal{F} \) after a finite number of steps.

Regardless of whether \( \text{abs}(\cdot) \) is finitary, a SLG\textsubscript{ABS} evaluation is complete with respect to a model that is restrained by the same abstraction function. If \( P \) is unsafe, SLG\textsubscript{ABS} may derive truth values that are not in \( \text{WFM}(\text{abs}(P), P) \), but that are in \( \text{WFM}(P) \). This occurs if SLG\textsubscript{ABS} derives a non-ground answer \( A \) for which \( A = \text{abs}(A) \), and for some atom \( A' \) in the ground instantiation of \( A \), \( A' \neq \text{abs}(A') \). In this case \( A' \) is undefined in \( \text{WFM}(\text{abs}(P), P) \), although \( A \) is true in the interpretation induced by the final forest of the SLG\textsubscript{ABS} evaluation (\( \mathcal{I}_{\mathcal{F}_{\text{fin}}} \)).

Theorem 4 Let \( \mathcal{E} \) be an SLG\textsubscript{ABS} evaluation of a query \( Q \) to a normal program \( P \) using abstraction function \( \text{abs}(\cdot) \), such that \( \mathcal{E} \) has a final forest \( \mathcal{F}_{\text{fin}} \). Then

\[
\text{WFM}(\text{abs}(P)|\mathcal{F}_{\text{fin}}) \subseteq \mathcal{I}_{\mathcal{F}_{\text{fin}}} \subseteq \text{WFM}(P)|\mathcal{F}_{\text{fin}}.
\]

Complexity of SLG\textsubscript{ABS}

The best currently known bound on worst case complexity for computing the well-founded semantics of a program \( P \) is \( \text{size}(P) \times |\text{atoms}(P)| \) (van Gelder, Ross, and Schlipf 1991). In order to relate the complexity of SLG\textsubscript{ABS} to this result, we extend the cost model of (Riguzzi and Swift 2013b).

The first aspect of our cost model, \( C_{\text{SLG-ABS}} \), addresses the fact that evaluations may terminate on ground programs that
are not finite. Let \( P \) be a ground (normal) program, and \( Q \) an atomic query to \( P \) (not necessarily ground). Then the atomic search space of \( Q \), \( P_Q \), consist of the union of all ground instantiations of \( Q \) in \( B_P \) together with all atoms reachable in the atom dependency graph of \( P \) from any ground instantiation of \( Q \). By Theorem 3 a SLG\_ABS evaluation \( \mathcal{E} \) of \( Q \) against \( P \) that uses a finitary abstraction function will produce a final forest \( \mathcal{F}_{\text{fin}} \) after a finite number of steps, and \( \mathcal{F}_{\text{fin}} \) will itself be finite. It is evident that the set of subgoals corresponding to trees in \( \mathcal{F}_{\text{fin}} \) (subgoals(\( \mathcal{F}_{\text{fin}} \)) is finite. Because \( \mathcal{F}_{\text{fin}} \) may contain non-ground subgoals, it is not the case that subgoals(\( \mathcal{F}_{\text{fin}} \)) \( \subseteq \) \( P_Q \); however if depth-k abstraction is used, it can be shown that \(|\text{subgoals}(\mathcal{F}_{\text{fin}})| \leq 2 \times |\text{atoms}(P_Q)|\).

Next, given the finite sequence \( \mathcal{E} \), we can construct the set of (ground) rules that were used in some PROGRAM CLAUSE RESOLUTION operation and denote this set as \( P_Q(\mathcal{E}) \). It is evident that \( P_Q(\mathcal{E}) \subseteq P_Q \subseteq P \), and that \( P_Q(\mathcal{E}) \) must always be finite. Define for a rule \( r \), size(\( r \)) as one plus the number of body literals in \( r \). Extending this, \( \text{size}(P_Q(\mathcal{E})) \) is defined as the sum of sizes of rules in \( P_Q(\mathcal{E}) \). \( C_{\text{SLGABS}} \) thus does not consider the size of terms within an atom or literal.

Finally, \( C_{\text{SLGABS}} \) determines the cost of each SLG\_ABS operation. Note, since the scope of an abstraction function is an atom, the cost of applying an abstraction function is constant in \( C_{\text{SLGABS}} \). Accordingly under \( C_{\text{SLGABS}} \) the NEW SUBGOAL, PROGRAM CLAUSE RESOLUTION, POSITIVE RETURN, NEGATIVE RETURN, DELAYING, and SIMPLIFICATION operations each affect one goal or delay literal and are considered constant time. The COMPLETION operation, however, applies to a set of subgoals \( S \) in a forest \( \mathcal{F} \) and its cost is proportional to the size of \( S \); in the worst case this is \(|\text{subgoals}(\mathcal{F})|\). Similarly, the ANSWER COMPLETION operation must determine an unsupported set of answers and its worst case is \( |\text{size}(P_Q(\mathcal{E}))| \).

The cost model \( C_{\text{SLGABS}} \) thus consists of

1. The definition of \( \text{subgoals}(\mathcal{F}) \) which is finite, and is \( O(|\text{atoms}(P)|) \) if \( |\text{atoms}(P)| \) is finite;
2. The definition of \( |\text{size}(P_Q(\mathcal{E}))| \) which finite and is \( O(|\text{size}(P)|) \) if \( |\text{size}(P)| \) is finite; and
3. Costs for each individual SLG\_ABS operation.

**Theorem 5** Let \( P \) be a ground normal program, \( Q \) a ground query, and \( \mathcal{E} \) a terminating SLG\_ABS evaluation of \( Q \) against \( P \) that uses depth-k abstraction, and with final forest \( \mathcal{F}_{\text{fin}} \). Then under the cost mode \( C_{\text{SLGABS}} \), the cost of \( \mathcal{E} \) is \( O(|\text{subgoals}(\mathcal{F}_{\text{fin}})| \times |\text{size}(P_Q(\mathcal{E}))|) \).

### Implementation, Performance and Scalability

SLG\_ABS is implemented using depth-k abstraction in version 3.3.7 of XSB (publically available at xsb.sourceforge.net), based in part on a prior implementation of subgoal abstraction. From the programmer’s perspective, depth-k abstraction is not used by default, but can be invoked using different values of \( k \) on a predicate basis. Answer abstraction is performed in the tabling engine of XSB, the SLG-WAM, during the check/insert step which checks whether an answer exists in a given table, and inserts the answer into the table if not. A counter maintains the current depth of the answer \( Ans \leftarrow \text{Delays} \) being traversed; if the depth of \( Ans \) is greater than \( k \) then the current subterm is replaced by a free (position) variable. In addition, the atom undefined\_abs, a reserved atom in XSB, is added to \( \text{Delays} \) if it is not already included, indicating that \( Ans \) is undefined. The overhead of answer abstraction is thus the cost of maintaining

![Figure 1: Final forest for the query r(X) to P\_abs-unsafe.](image-url)
the depth-counter, along with that of copying $\text{undefined}_{\text{abs}}$ into $\text{Delays}$ if the depth bound is exceeded.

If no answer abstraction function is specified (so that answers will not be abstracted) the overhead consists solely of the cost of maintaining the the depth counter within the answer check/insert operation. For various forms of linear recursion, we measured this overhead at $0\% - 4\%$ based on the ratio of answers to subgoals in a given benchmark.

A series of independent studies have shown XSB to be highly scalable (OpenRuleBench 2011). In addition, recent work with trace-based analysis in XSB has performed sophisticated analysis on trace logs with $10^7$ to $10^8$ and more events, where each event corresponds to a Prolog fact that is dynamically loaded for the analysis. This scalability has not been affected by the extension to SLG$_{ABS}$ in the SLG-WAM.

**Discussion**

This paper has shown how radially restrained well-founded models of a program approximate the well-founded model in a clear manner (Theorem 2). Queries to these models terminate correctly (Theorems 3 and 4) with low abstract complexity (Theorem 5). Tabled resolution for restrained models can be implemented with low overhead on performance, without impacting the scalability of query evaluation.

Using these results, work is ongoing to study how bounded rationality is best exploited for practical knowledge representation. Abstractions based on norms that distinguish between lists and other function symbols are being used within the Silk project (silk.semwebcentral.org); while abstractions based on size rather than depth may also prove useful. Analysis of these abstraction functions to estimate the maximum number of answers based on a non-ground input program, is underway. In addition, justification mechanisms are being developed to inform users whether an atom was undefined because of restraint, an unsafe use of negation, or simply through lack of stratified negation.

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References


