Resource Allocation and Multiagent Policy Formulation for Resource-Limited Agents Under Uncertainty

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Abstract. The problem of optimal policy formulation for teams of resource-limited agents in stochastic environments is composed of two strongly coupled subproblems: a resource allocation problem and a policy optimization problem, both of which have individually received significant amount of attention. We show how to combine the two problems into a single constrained optimization problem that yields optimal resource allocations and policies that are optimal under these allocations. We model the stochastic environment as a multiagent Markov decision process, with social welfare of the group as the optimization criterion. We augment the standard MDP framework with constraints that ensure that the shared resource limitations are satisfied and formulate a constrained stochastic policy optimization problem that yields optimal policies among the class of realizable ones given the resource limitations. This work focuses on discrete operationalization resources that determine the actuating capabilities of the agents by defining the sets of actions available to them. We show that the problem of finding optimal policies under such constraints is NP-hard and present a solution algorithm based on mixed integer programming to solve the corresponding optimization problems.

1 Introduction

We address the problem of finding optimal policies for teams of resource-limited autonomous agents that operate in stochastic environments. While various aspects of this problem have received significant amount of attention [1–6], there has been little effort in addressing the combined problem of deciding how the limited shared resources should be distributed between the agents and what policies they should adopt, such that the social welfare of the team is maximized. Notice that in this problem formulation, figuring out the value of a particular allocation requires one to solve a stochastic policy optimization problem. Hence, the resource allocation and the policy optimization problems are very closely coupled. The contributions of the work we present here are that we formally analyze the complexity of this problem and develop a solution method that can capitalize on efficient methods of solving mixed integer programs.

We begin by giving a very broad, high-level description of the problem and presenting a simple example of a domain where such problems arise. We then present a formal description of our model and the problem formulation as well as an analysis of

the complexity of this optimization problem and the structure of the solutions. The last sections of the paper describe our method for solving this problem and present some empirical results.

1.1 Motivating Example

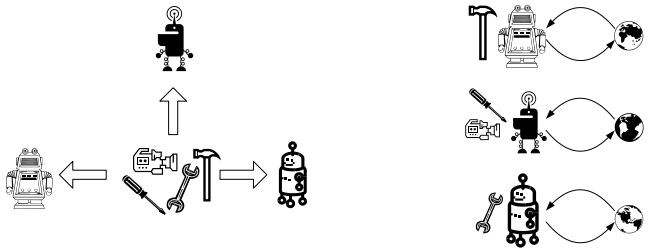
Imagine a group of agents that are operating autonomously – for example, a group of rovers performing a scientific mission on a remote planet. There is a clear need for coordination and task allocation among the agents in order for them to perform their mission efficiently. For instance, if the mission involves taking measurements of the soil in one location and doing a video survey of another area, it might be beneficial to assign one rover to do soil sampling, and another to do the video survey. However, the agents typically need different equipment to carry out different tasks, and while it is sometimes feasible to design and build an agent for a certain task, often if is more effective to create a general-purpose agent that can be outfitted with different equipment depending on the task at hand. In particular, in our rover example, there might be a base station that is used as a centralized location to store consumable execution resources (fuel, energy, etc), as well as equipment (video cameras, extra batteries, etc.) that can be used to outfit the rovers for various tasks. It is certainly natural to assume that these resources are limited. Thus, a problem of efficient allocation of the shared resources arises.

However, the resource allocation problem is complicated by the fact that it is often hard to calculate the exact utility of a particular assignment of the resources. Indeed, agents operating in complex environments are not able to perfectly and deterministically reason about the effects of their actions, and therefore it is necessary to adopt a probabilistic model of their interactions with the environment. In fact, an agent that adopts a deterministic model of the environment and does not have contingency plans might find itself acting rather poorly. For instance, it has been estimated that the 1997 Mars Pathfinder, which did not take the uncertainty of the environment into account, spent between 40% to 75% of its time doing nothing due to plan failures [7]. Therefore, in order to determine the value of a particular resource allocation for agents operating in stochastic environments, it is necessary to solve a stochastic optimization problem.

1.2 General Problem Description

More generally, it is ofter the case that an agent has many capabilities that are all in principle available to it, but not all combinations are realizable within the architectural limitations, because choosing to enable some of the capabilities might usurp resources needed to enable others. In other words, a particular policy might not be *operational* because the agent's architecture does not support the combination of capabilities required for that policy. If this is the case, we say that the agent exhibits *operationalization* constraints.

At a high level, we model the situation described above as follows (illustrated in Figure 1). The agents have a set of actions that are potentially executable, but each action requires a certain combination of resources. The amount of these shared resources is limited. Furthermore, each agent has constraints as to what resources it can make use of



(a) (b)

Fig. 1. A resource allocation and stochastic planning problem. Once the shared operationalization resources are distributed among the agents (a), they proceed to execute the best realizable policies (b).

(for example, what equipment it can be outfitted with). In our model, execution begins with a distribution of the shared resources among the agents (Figure 1a). Any resulting resource allocation must obey the constraints that no shared resource is over-utilized, i.e. the amount of all resources that are assigned to the agents does not exceed the total available amount. Furthermore, the assignment must satisfy the local constraints of the agents as to the resources that they can use. For example, it is useless (and thus essentially invalid) to assign to an agent more equipment that it can carry. Once the shared resources are distributed among the agents, they should use these resources to carry out their policies (Figure 1b) in such a way that the social welfare of the group (sum of individual rewards) is maximized.

2 The Model

Before we describe our world model in more detail, let us note that although it is very easy to adapt our model and solution algorithms to a scenario that involves consumable execution resources (ex. fuel, time) in addition to discrete operationalization resources (ex. equipment to outfit the agents), we do not model the former in this paper. This is done in the interest of space and for ease of exposition. It turns out that including such continuous consumable resources in the model does not add to the complexity of the problem, but introduces some subtleties to the optimization and has an effect on the structure of the optimal policies [8].

The stochastic properties of the environments in the problems that we address in this work lead us to adopt the Markov model as the underlying formalism. In particular, we use the stationary, discrete-time Markov model with finite state and action spaces [4]. The choice is due to the fact that MDPs provide a well-studied and simple, yet a very expressive, model of the world. This section briefly describes standard unconstrained Markov decision processes and discusses the assumptions that are specific to the problems that we focus on in this work.

2.1 Markov Decision Processes

A classical unconstrained single-agent MDP can be defined as a tuple $\langle \mathcal{S}, \mathcal{A}, \mathbf{P}, \mathbf{R} \rangle$, where:

- $\mathcal{S} = \{i\}$ is a finite set of states.
- $\mathcal{A} = \{a\}$ is a finite set of actions.
- $\mathbf{P} = [p_{iaj}] : \mathcal{A} \times \mathcal{S} \times \mathcal{S} \rightarrow [0, 1]$ defines the transition function. The probability that the agent goes to state j if it executes action a in state i is p_{iaj}.
- $\mathbf{R} = [r_{ia}] : \mathcal{S} \times \mathcal{A} \rightarrow \Re$ defines the rewards. The agent gets a reward of r_{ia} for executing action a in state i.

A policy is defined as a procedure for selecting an action in each state. A policy is said to be *stationary* if it does not depend on time, but only on the current state, i.e. the same procedure for selecting an action is performed every time the agent encounters a particular state. A *deterministic* policy always chooses the same action for a state, as opposed to a *randomized* policy, which chooses actions according to some probability distribution over the set of actions. The term *pure* is used to refer to stationary deterministic policies.

A randomized Markov policy $\boldsymbol{\pi}$, can be described as a mapping of states to probability distributions over actions, or equivalently, as a mapping of state-action pairs to probability values: $\boldsymbol{\pi} = [\pi_{ia}] : \mathcal{S} \times \mathcal{A} \rightarrow [0, 1]$; π_{ia} defines the probability of executing action a, given that the agent is in state i. We assume that an agent must execute an action (a noop is considered a trivial action) in every state, thus $\sum_a \pi_{ia} = 1$.

A pure policy can be viewed as a degenerate case of a randomized policy, for which there is only one action for each state that has a nonzero probability of being executed (that probability is obviously 1).

Clearly, the total probability of transitioning out of a state, given a particular action, cannot be greater than 1, i.e. $\sum_j p_{iaj} \leq 1$. As discussed below, often we are are actually interested in domains where there exist states for which $\sum_j p_{iaj} < 1$.

If, at time 0, the agent has an initial probability distribution $\boldsymbol{\alpha} = [\alpha_i]$ over the state space, and the system obeys the Markov assumption (namely that the transition probabilities depend only on the current state and the chosen action), the system's trajectory (defined as a sequence of probability distributions on states) will be as follows:

$$\boldsymbol{\rho}(t+1) = \widetilde{\mathbf{P}}\boldsymbol{\rho}(t), \quad \boldsymbol{\rho}(0) = \boldsymbol{\alpha}, \tag{1}$$

where $\boldsymbol{\rho}(t) = [\rho_i(t)]$ is the probability distribution of the system at time t ($\rho(t)_i$ is the probability of being in state i at time t), and $\widetilde{\mathbf{P}} = [\widetilde{p}_{ij}]$ is the transition probability matrix implied by the policy ($\widetilde{p}_{ij} = \sum_a p_{iaj}\pi_{ia}$).

2.2 Assumptions

Typically, Markov decision problems are divided into two categories: *finite-horizon* problems, where the total number of steps that the agent spends in the system is finite and is known a priori, and *infinite-horizon* problems, where the agent is assumed to stay in the system forever (see, for example, [4]).

In this work we focus on dynamic real-time domains, where agents have tasks to accomplish. This leads us to make a slightly different (although, certainly, not novel) assumption about how much time the agent spends executing its policy. We assume that there is no predefined number of steps that the agent spends in the system, but that optimal policies always yield *transient* Markov processes [9]. A policy is said to yield a transient Markov process if the agent executing that policy will eventually leave the corresponding Markov chain, after spending a finite number of time steps in it. Given a finite state space, this assumption implies that the Markov chain corresponding to the optimal policy has no recurrent states (states that have a nonzero probability of being visited infinitely many times) or, in other words: $\lim_{t\to\infty} \rho_i(t) = 0$. This means that there has to be some "leakage" of probability out of the system, i.e. there have to exist some states $\{i\}$ for which $\sum_j \sum_a p_{ij}^a < 1$.

We also assume that the rewards that an agent receives while executing a policy are bounded. Given these assumptions about bounded rewards and the transient nature of our problems, the most natural policy evaluation function to adopt is the expected total reward:

$$V(\boldsymbol{\pi}, \boldsymbol{\alpha}) = \sum_{t=0}^{T} \sum_i \boldsymbol{\rho}_i^t \sum_a \pi_{ia} r_{ia}, \tag{2}$$

where T is the number of steps during which the agent accumulates utility. For a transient system with bounded rewards, the above sum converges for any T.

If the system obeys the Markov assumption and, as a result, follows the trajectory in (eq. 1), the value of a policy can be expressed in terms of the initial probability distribution, transition probabilities, and rewards as follows:

$$V(\boldsymbol{\pi}, \boldsymbol{\alpha}) = \sum_{t}^{\infty} \mathbf{R}\boldsymbol{\rho}(t) = \sum_{t}^{\infty} \mathbf{R}\widetilde{\mathbf{P}}^t \boldsymbol{\alpha}, \tag{3}$$

which, under our assumptions, becomes:

$$V(\boldsymbol{\pi}, \boldsymbol{\alpha}) = \mathbf{R}(\mathbf{I} - \widetilde{\mathbf{P}})^{-1} \boldsymbol{\alpha} \tag{4}$$

It is clear that the value of a policy depends on the initial state probability distribution $\boldsymbol{\alpha}$. Moreover, in general, the relative order of two policies can change depending on the initial probability distributions, i.e. $\exists \boldsymbol{\pi}, \boldsymbol{\pi}', \boldsymbol{\alpha}, \boldsymbol{\alpha}' : V(\boldsymbol{\pi}, \boldsymbol{\alpha}) > V(\boldsymbol{\pi}', \boldsymbol{\alpha})$, and $V(\boldsymbol{\pi}, \boldsymbol{\alpha}') < V(\boldsymbol{\pi}', \boldsymbol{\alpha}')$. However, often, there exist policies that are optimal for *any* initial probability distribution, i.e. $V(\boldsymbol{\pi}^*, \boldsymbol{\alpha}) \geq V(\boldsymbol{\pi}, \boldsymbol{\alpha})\ \forall \boldsymbol{\pi}, \boldsymbol{\alpha}$. These policies are the ones that are commonly called "optimal" in the unconstrained MDP literature and are typically computed via dynamic programming, based on Bellman optimality equations [10]. We refer to these policies as *uniformly optimal* (using the terminology from [5]). These uniformly optimal policies $\boldsymbol{\pi}^*$ always produce a history of states that is at least as good as a history produced by any other policy, regardless of the initial conditions. Therefore, if uniformly optimal policies exist, it is sufficient to compute a single uniformly optimal policy and use it for all instances of the problem with arbitrary initial probability distributions.

However, as it turns out, uniformly optimal policies do not always exist for constrained problems that involve limited resources (we will prove this statement below).

We are, therefore, interested in finding optimal policies for a given initial probability distribution $\boldsymbol{\alpha}$.

Let us note that, although in this work we focus on transient systems with the total expected reward optimization criterion, our model, the complexity results, and the solution algorithm can be easily adapted to other commonly-used Markov models (finite horizon, infinite horizon with discounted or per-unit rewards).

3 Problem Description

3.1 Multi-Agent Markov Decision Processes

Let us now consider a multiagent environment with a set of n agents $\mathcal{M} = \{m\}$ ($|\mathcal{M}| = n$), each of whom has its own set of states $\mathcal{S}_m = \{i_m\}$ and actions $\mathcal{A}_m = \{a_m\}$. Without any loss of generality, we can assume that the state and action spaces are equal ($\mathcal{S}_m = \mathcal{S}_{m'}, \quad \mathcal{A}_m = \mathcal{A}_{m'} \quad \forall m, m' \in \mathcal{M}$).

In general, for a multiagent MDP, we have to define a new state space that is the cross-product of the state spaces of all agents: $\mathcal{S}(\mathcal{M}) = \mathcal{S}^n$, and a new action space that is the cross-product of the actions spaces of all agents: $\mathcal{A}(\mathcal{M}) = \mathcal{A}^n$. The transition and reward functions are defined on the new state and action space, i.e. $\mathbf{P}(\mathcal{M}) : \mathcal{S}^n \times \mathcal{S}^n \times \mathcal{A}^n \rightarrow [0, 1]$, and $\mathbf{R}(\mathcal{M}) : \mathcal{S}^n \times \mathcal{A}^n \rightarrow \Re$. However, there is a large subclass of multiagent domains where the agents' rewards and transition functions are independent of each other, i.e. such problems are completely separable if there are no shared resources involved.

It turns out that our analysis and solution methodology is directly applicable to both the general model formulation as well as the model that assumes that agents have independent reward and transition functions. Therefore, in order to simplify notation, we assume that once the shared resources are distributed, the agents operate completely independently of each other. In other words, each agent has its own independent reward and transition functions defined on $\mathcal{S}$ and $\mathcal{A}$. It is important to note that although our solution method is directly applicable to both models, the sizes of the two problems differ greatly. Without the independence assumption, the number of optimization variables in our optimization program (section 5) would be $\prod_m |\mathcal{S}_m| \prod_m |\mathcal{A}_m|$, whereas the independence assumption brings the number of variables down to $\sum_m |\mathcal{S}_m||\mathcal{A}_m| \approx n|\mathcal{S}||\mathcal{A}|$, which is exponentially smaller.

Under the above independence assumption, a *joint policy* of the group is simply the set of single-agent policies of all agents, i.e. $\boldsymbol{\pi}(\mathcal{M}) = [\boldsymbol{\pi}^m] = [\pi^m_{ia}]$.

3.2 Problem Formulation

We can now define the problem of multiagent policy optimization under limited shared resources. Let us say that there are several shared resources, and that every action of each agent requires some subset of these resources. Furthermore, all resources have costs associated with them and agents have upper bounds on the costs of resources that can be allocated to them. For example, a problem might involve shared equipment (ex. tools) that enables agents to execute various actions, but each unit of equipment has

some costs associated with it (ex. weight), and the agents have upper bounds on how much weight they can carry.

Under these conditions, we can formulate the multiagent optimization problem as a tuple $\langle \mathcal{S}, \mathcal{A}, \mathbf{P}, \mathbf{R}, \mathbf{C}, \widehat{\mathbf{C}}, \mathbf{Q}, \widehat{\mathbf{Q}}, \boldsymbol{\alpha} \rangle$, where:

- $\mathcal{S} = \{i\}$ is a finite set of states.
- $\mathcal{A} = \{a\}$ is a finite set of actions.
- $\mathbf{P} = [\mathbf{P}^m] = [p^m_{iaj}] : \mathcal{A} \times \mathcal{S} \times \mathcal{S} \rightarrow [0, 1]$ defines the transition function for agent m. The probability that agent m goes to state j if it executes action a in state i is p^m_{iaj}.
- $\mathbf{R} = [\mathbf{R}^m] = [r^m_{ia}] : \mathcal{S} \times \mathcal{A} \rightarrow \Re$ defines the rewards that agent m receives. Agent m gets a reward of r^m_{ia} for executing action a in state i.
- $\mathbf{C} = [\mathbf{C}^m] = [c^m_{ak}]$ defines action resource requirements. Agent m needs to have c^m_{ak} units of resource k to be able to execute action a.
- $\widehat{\mathbf{C}} = [\hat{c}_k]$ defines the total amounts of shared resources that are available to the group, i.e. there are $\hat{c}_k$ units of resource k available to the agents.
- $\mathbf{Q} = [q^m_{kl}]$ defines the costs (weight, money, etc) of each resource. The cost of type l of a unit of resource k is given by q^m_{kl}.
- $\widehat{\mathbf{Q}} = [\hat{q}^m_l]$ defines the upper bounds on how much of the costs the agent can incur (ex. how much weight the agent can hold or how much money it can spend). Agent m cannot exceed $\hat{q}^m_l$ units of cost of type l.
- $\boldsymbol{\alpha} = [\boldsymbol{\alpha}^m] = [\alpha^m_i]$ is the initial probability distribution. The probability that agent m starts in state i is α^m_i.

Without any loss of generality, we assume that the action space $\mathcal{A}$ and the state space $\mathcal{S}$ are the same for all agents.

The goal of the optimization problem is to find a joint policy $\boldsymbol{\pi}(\mathcal{M})$ that yields the highest expected reward, under the conditions that the shared resources are not over-utilized and that no agent is assigned more resources that it can hold. In other words, we have to solve the following (abstract) math program:

$$\max V(\boldsymbol{\pi}, \boldsymbol{\alpha}) \left| \begin{array}{l} \sum_m \sum_a c^m_{ak} \theta(\sum_i \pi^m_{ia}) \leq \hat{c}_k, \\ \sum_k q_{kl} \sum_a c^m_{ak} \theta(\sum_i \pi^m_{ia}) \leq \hat{q}^m_l, \end{array} \right. \tag{5}$$

where θ is a "step" function of a non-negative argument, defined as:

$$\theta(z) = \begin{cases} 0 & z = 0 \\ 1 & z > 0 \end{cases}$$

The first constraint in (eq. 5) means that the total amounts of resources that are needed by all agents do not exceed the total amounts that are available, and the second one has the meaning that the cost of the resources assigned to an agent does not exceed its cost bounds. Indeed, agent m incurs the cost c^m_{ak} (or, alternatively, requires that much) of resource k only if it wants to include action a for any states in its policy, which happens if and only if the agent's policy dictates a nonzero probability of executing action a in any state i, i.e. $\pi^m_{ia} > 0$, in which case $\theta(\sum_i \pi^m_{ia}) = 1$.

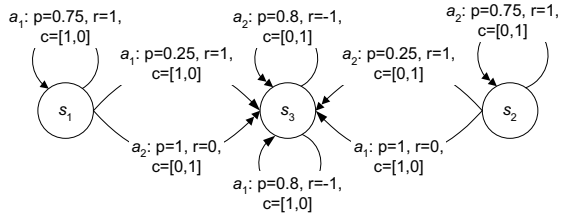


Fig. 2. Uniformly-optimal policies do not always exist for constrained problems. Transition probabilities, rewards, and action costs are shown on the diagram.

4 Problem Properties

4.1 Policy Structure

In this section we show some properties of the optimal policies. We begin by demonstrating that uniformly-optimal policies (optimal for any initial distribution) do not always exist for constrained problems. This result is well known for classical constrained MDPs [5, 4] where constraints are imposed on the total expected costs that are proportional to the expected number of times the corresponding actions are executed. We now establish this result for problems with operationalization constraints (eq. 5) for which the costs are incurred by the agents when they include an action in their policy, regardless of how many times the action is actually executed (the costs are interpreted as the amounts of the shared resources that are required to enable the action). The following (fairly obvious) proposition shows the validity of the claim by example.

Proposition 1 *There do not always exist uniformly-optimal solutions to Markov decision problems with operationalization constraints (eq. 5).*

Proof: We present an example (Figure 2), which makes the above statement clear. Let us consider a problem with two identical agents ($m = \{1, 2\}$), three states $\{s_1, s_2, s_3\}$, two actions $\{a_1, a_2\}$, and two shared resource types ($k = \{1, 2\}$). The rewards (r), transition probabilities (p), and action costs (c) for all actions are shown in Figure 2. The resource costs (requirements) for action a_1 are $[1, 0]$, i.e. it needs one unit of the first resource and does not use the second resource at all. Action a_2 is exactly the opposite. States s_1 and s_2 are "good", because the agents can receive positive rewards ($r = 1$) there, and state s_3 is "bad", since the reward there is negative ($r = -1$).

The obvious unconstrained optimal policy for both agents is to execute action a_1 in state s_1, action a_2 in state s_2 and either action (or a probabilistic mixture of the two) in state s_3. However, in order for both agents to be able to execute this policy, they each need to have one unit of each of the resource types.

Let us now assume that there is only one unit of each resource available ($\hat{c} = [1, 1]$), and let us show that there does not exist a policy that is optimal for all initial probability distributions $\boldsymbol{\alpha}$.

Consider the situation where the first agent starts in state s_1 and the second agent starts in state s_2, i.e. $\boldsymbol{\alpha}^1 = [1, 0],\ \boldsymbol{\alpha}^2 = [0, 1]$. Then, the obvious unique optimal joint

policy that satisfies the resource constraints is $\boldsymbol{\pi}^1 = [(1,0),(1,0),(1,0)]$ and $\boldsymbol{\pi}^2 = [(0,1),(0,1),(0,1)]$. [1] The resource cost of this joint policy is $[1,1]$, which satisfies the constraints on the total use of the shared resources. The value (total expected reward) of that policy is zero ($V(\boldsymbol{\alpha},\boldsymbol{\pi}) = 0$), since each agent, on average, receives the positive reward (+1) four times before it transitions to state s_3, where its expected payoff is -4, yielding a total expected value of zero. This policy is clearly the unique optimum for the given initial conditions, because any other policy would either not satisfy the constraints or yield a lower (negative) payoff.

If we consider a problem with the reversed initial conditions $\boldsymbol{\alpha}'$ where the first agent starts in state s_2 and the second agents starts in state s_1, the policy described above would immediately take both agents to state s_3 and would yield a total expected reward of -8. However, clearly, there also exists a policy that yields a zero expected reward for this initial distribution. Thus, our policy $\boldsymbol{\pi}$ is the unique optimal solution to the problem with the initial distribution $\boldsymbol{\alpha}$ and is a suboptimal solution to the problem with the initial distribution $\boldsymbol{\alpha}'$.

We have therefore constructed an example for which no uniformly-optimal policy exists. ■

4.2 Complexity

In this section we study the computational complexity of the multiagent optimization problem introduced earlier. We begin by defining a corresponding decision problem, which we label M-OPER-CMDP (a multiagent constrained MDP with operationalization constraints):

> *Given an instance of a multiagent MDP with shared operationalization resources $\langle \mathcal{S}, \mathcal{A}, \mathbf{P}, \mathbf{R}, \mathbf{C}, \widehat{\mathbf{C}}, \mathbf{Q}, \widehat{\mathbf{Q}}, \boldsymbol{\alpha} \rangle$ and a rational number V, does there exist a multiagent policy $\boldsymbol{\pi}$, whose expected total reward, given $\boldsymbol{\alpha}$, equals or exceeds V?*

The following result characterizes the complexity of this decision problem. It assumes that pure (stationary deterministic history-independent) policies are optimal for this class of problems. This can be shown via the same arguments as in the case of standard unconstrained MDPs, but we omit the proof here in the interest of space. Intuitively, there is no need to use randomized policies, because including an action in a policy incurs the same resource costs, regardless of the probability of executing that action (or the expected number of times the action will be executed).

Theorem 1 M-OPER-CMDP *is NP-complete.*

Proof: The presence of M-OPER-CMDP in NP is obvious. Clearly, one can always guess a pure joint policy, verify that it satisfies the shared resource constraints, and calculate its expected total reward in polynomial time (the latter can be done by solving the standard system of linear Markov equations on the values of all states [4]).

[1] Here and below we use round parentheses as a notational convenience to group the values that refer to one state, i.e. $\boldsymbol{\pi} = [(\pi_{11},\pi_{12}),(\pi_{21},\pi_{22}),(\pi_{31},\pi_{32})]$.

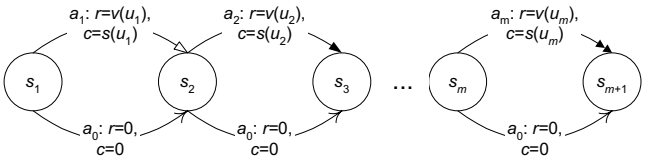


Fig. 3. Reduction to of KNAPSACK to M-OPER-CMDP. All transitions are deterministic.

In order to show NP-completeness of M-OPER-CMDP, we will reduce KNAPSACK to a single-agent instance of M-OPER-CMDP with one resource. Recall that KNAPSACK asks whether, for a given set of items $u \in U$, each of which has a cost $t(u)$ and a value $v(u)$, there exists a subset $U' \subseteq U$ such that the total value of all items in U' is no less than some constant W, and the total cost of is no greater than another constant B, i.e. $\sum_{u \in U'} t(u) \leq B, \sum_{u \in U'} v(u) \geq W$. KNAPSACK is known to be NP-complete [11]. Therefore, if we show that any instance of KNAPSACK can be reduced to M-OPER-CMDP, we would show that M-OPER-CMDP is also NP-complete. The reduction is illustrated in Figure 3 and proceeds as follows.

Given an instance of KNAPSACK with $|U| = m$, let us number all items as $u_i,\ i \in [1, m]$ as a notational convenience. For such an instance of KNAPSACK, we create a MDP with $m + 1$ states $\{s_1, s_2, \ldots s_{m+1}\}$. For every item u_i in KNAPSACK, we define an action a_i with reward $v(u_i)$ and cost $t(u_i)$. We also define a "null" action a_0 with zero cost and zero reward. Furthermore, we define a deterministic transition function on these states as follows. Every state $s_i,\ i \in [1, m]$ has two transitions from it – corresponding to actions a_i and a_0. Both actions lead to state s_{i+1} with certainty, but a_i gives the agent a reward of $v(u_i)$, while action a_0 gives a reward of zero. State s_{m+1} is absorbing and has no transitions leading from it.

In order to complete the construction of M-OPER-CMDP, we set the initial distribution $\boldsymbol{\alpha} = [1, 0, \ldots]$, so that the agent starts in state s_1 with probability 1. We also define the decision parameter $V = W$ and the total amount of the single resource $\hat{c}_1 = B$. We make the cost constraints non-binding by setting $q_{kl} = 0$ and $\hat{q}_l = \infty$.

The above construction basically allows the agent to choose action a_i or a_0 at every state s_i. Choosing action a_i is equivalent to putting item u_i into the knapsack, while action a_0 corresponds to the choice of not including u_i in the knapsack. Therefore, it is clear that there exists a policy that has the expected payoff no less than $V = W$ and uses no more than $\hat{c}_1 = B$ of the shared resource if and only if there exists a solution to the original instance of KNAPSACK. ■

Note that we have formulated and proven Theorem 1 for a transient processes, because we focus on such processes in this work. However, the result also holds for MDPs with a finite-horizon, or an infinite horizon with total discounted or average per-step rewards. Indeed, the complexity proofs for all of these flavors of MDPs are almost identical and can be done via simple modifications of the above reduction.

5 Solution Algorithm

In this section we present an algorithm for solving the optimization program (eq. 5). However, before we describe our algorithm, let us briefly review the standard linear programming approach [12, 9] to solving Markov decision processes, which serves as the basis for our method.

A common method for finding a solution to a transient single-agent unconstrained MDP is by solving the following linear program:

$$\max \sum_i \sum_a x_{ia} r_{ia}$$

subject to the constraints:

$$\sum_a x_{ja} - \sum_i \sum_a x_{ia} p_{iaj} = \alpha_j, \quad x_{ia} \geq 0,$$

or, equivalently:[2]

$$\max \sum_i \sum_a x_{ia} r_{ia} \left| \sum_i \sum_a (\delta_{ij} - p_{iaj}) x_{ia} = \alpha_j, \right. \tag{6}$$

where δ_{ij} is the Kronecker delta, defined as $\delta_{ij} = 1 \iff i = j$. The optimization variables $\mathbf{x} = [x_{ia}]$ are often referred to as the *occupancy measure* of a policy, and x_{ia} can be interpreted as the expected number of times action a is executed in state i. The constraints in (eq. 6) just represent the conservation of probability and have nothing to do with external constraints imposed on the problem. That is, the expected number of times that state j is visited less the expected number of times that j is entered across all state-action pairs should equal the expected number of times of starting in state j (i.e. initial probability of being in j).

A policy $\boldsymbol{\pi}$ can be computed from $\mathbf{x}$ simply as:

$$\pi_{ia} = \frac{x_{ia}}{\sum_a x_{ia}} = \frac{x_{ia}}{x_i} \tag{7}$$

Under our independence assumptions, we can analogously construct a linear program for an unconstrained multiagent MDP with the total expected reward as the optimization criterion:

$$\max \sum_m \sum_i \sum_a x^m_{ia} r^m_{ia} \left| \sum_i \sum_a (\delta_{ij} - p^m_{iaj}) x^m_{ia} = \alpha^m_j, \right. \tag{8}$$

where x^m_{ia} is the expected number of times agent m executes action a in state i. A solution to this LP yields optimal policies for the unconstrained multiagent problem. It is easy to see that the above LP is completely separable and could have been written as $|\mathcal{M}|$ single-agent LPs. We, however, are interested in solving the constrained optimization problem that we have earlier written in an abstract form as (eq. 5). Let us

[2] From now on we will omit the $x_{ia} \geq 0$ constraint for brevity.

now rewrite the program (eq. 5) in the occupancy measure coordinates $\mathbf{x}$. Adding the constraints from (eq. 5) to (eq. 8), and noticing that $\theta(\sum_i \pi_{ia}^m) = \theta(\sum_i x_{ia}^m)$, we get the following optimization problem in $\mathbf{x}$:

$$\max \sum_m \sum_i \sum_a x_{ia}^m r_{ia}^m \left| \begin{array}{l} \sum_i \sum_a (\delta_{ij} - p_{iaj}^m) x_{ia}^m = \alpha_j^m, \\ \sum_m \sum_a c_{ak}^m \theta(\sum_i x_{ia}^m) \leq \hat{c}_k, \\ \sum_k q_{kl} \sum_a c_{ak}^m \theta(\sum_i x_{ia}^m) \leq \hat{q}_l^m, \end{array} \right. \tag{9}$$

If we could solve this math program, we would be done. The big problem with this program is, of course, that it involves the "step" function θ, which makes the constraints nonlinear (and actually discontinuous at zero). Consequently, our goal is to rewrite the program (eq. 9) in a more manageable form. However, we can abandon the idea of trying to write it as a linear program, because LPs are solvable in polynomial time, and we have shown that our problem is NP-hard. Instead, we are going to reformulate the problem as a mixed integer program (MIP) (also NP-hard). The ability to formulate the constrained policy optimization problem as a MIP allows us to make use of a wide variety of highly optimized algorithms and tools for solving integer programs.

Let us augment the original optimization variables $\mathbf{x}$ with a set of binary variables $\mathbf{\Delta} = [\Delta_a^m]$, where $\Delta_a^m = \theta(\sum_i x_{ia}^m)$. In other words, Δ_a^m is an indicator variable that shows whether agent m plans to use action a in at least one state. Using $\mathbf{\Delta}$, we can rewrite the resource constraints in (eq. 9) as

$$\sum_m \sum_a c_{ak}^m \Delta_a^m \leq \hat{c}_k, \quad \sum_k q_{kl} \sum_a c_{ak}^m \Delta_a^m \leq \hat{q}_l^m, \tag{10}$$

which are linear in $\mathbf{\Delta}$. Now, if we could "synchronize" $\mathbf{x}$ and $\mathbf{\Delta}$ to preserve their intended interpretation via a linear function, we would have a linear mixed integer program, which is the goal that we have set forth in this section. The problem is, of course, that the relationship between Δ_a^m and x_{ia}^m is nonlinear:

$$\Delta_a^m = \begin{cases} 0, & \text{if } \sum_i x_{ia}^m = 0 \\ 1, & \text{if } \sum_i x_{ia}^m > 0 \end{cases} \tag{11}$$

Note that this is exactly the step function that we wanted to get rid of in the first place. However, we can capture the essence of the relationship between $\mathbf{x}$ and $\mathbf{\Delta}$ with a linear function by using a couple of "tricks".

First, we need to normalize our occupancy measure ($\mathbf{x}$) such that $\sum_i x_{ia} \in [0, 1]$. Let us define a new normalized occupancy measure $\mathbf{y}$ as:

$$y_{ia}^m = \frac{x_{ia}^m}{X}, \tag{12}$$

where $X \geq \sup \sum_a x_{ia}^m$ is some constant finite upper bound on $\sum_i x_{ia}^m$, which exists for any transient MDP and can be computed in polynomial time. Indeed, one simple

way to do this is to replace the expected reward in the objective function in the standard unconstrained LP (eq. 6) with $\sum_m \sum_i \sum_a x_{ia}^m$, solve this LP in polynomial time and let X equal the resulting value of this objective function.

Given the normalized occupancy measure, we can then capture the essence of the relationship between $\mathbf{x}$ and $\mathbf{\Delta}$ via a linear constraint:

$$\sum_i y_{ia}^m \leq \Delta_a^m \tag{13}$$

Clearly, if $\sum_i y_{ia}^m > 0$, the above constraint forces the corresponding Δ_a^m to be 1, which is in accordance with (eq. 11). On the other hand, if $\sum_i y_{ia}^m = 0$, the above constraint will hold for both $\Delta_a^m = 0$ and $\Delta_a^m = 1$, which does not quite satisfy (eq. 11). However, it turns out that this is not a problem for the following reason. If some $\Delta_a^m = 1$, even if the corresponding $\sum_i y_{ia}^m = 0$ (which is allowed by constraint (eq. 11)), the worst that can happen is that the constraint (eq. 10) on the shared resources becomes unnecessarily broken. This might seem problematic but, in fact, it is not, since the important thing is that another (feasible) solution with the offending deltas corrected always exists and has the same value of the objective function. Basically, the above condition has no false positives, but can have false negatives, which are not lethal. This means that any complete algorithm for solving MIPs will always find the optimal deterministic policy that satisfies the constraints (if one exists).

To summarize, the problem of finding optimal policies under operationalization constraints can be formulated as the following MIP:

$$\max \sum_m \sum_i \sum_a y_{ia}^m r_{ia}^m \left| \begin{array}{l} \sum_i \sum_a (\delta_{ij} - p_{iaj}^m) y_{ia}^m = \frac{\alpha_j^m}{X}, \\ \sum_m \sum_a c_{ak}^m \Delta_a^m \leq \hat{c}_k, \\ \sum_k q_{kl} \sum_a c_{ak}^m \Delta_a^m \leq \hat{q}_l^m, \\ \sum_i y_{ia}^m \leq \Delta_a^m, \\ y_{ia}^m \geq 0, \quad \Delta_a^m \in \{0, 1\} \end{array} \right. \tag{14}$$

As mentioned earlier, even though such programs are, in general, NP-hard, there is a wide variety of very efficient algorithms and tools for doing so (see, for example, [13] and references therein). Therefore, one of the benefits of reducing the optimization problem to MIP is that it allows us to make use of the existing highly efficient tools.

As we mentioned earlier, our solution algorithm does not rely on the independence of agents' reward and transition functions. In fact, a fully-observable multiagent MDP is equivalent to a large single-agent problem, and we can easily model the constraints on the shared resources by simply assigning appropriate costs to the joint actions. We can then solve the resulting problem in exactly the same way as we did for independent agents.

6 Empirical Validation

We have implemented the algorithm from the previous section and have run it on a series of test problems to see how it behaved. Our main goals have been to see how well the algorithm scales and also how it behaves as resource constraints are tightened or

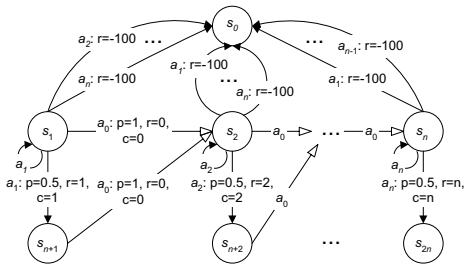
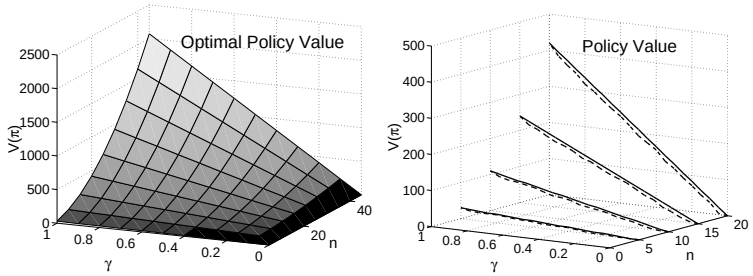


Fig. 4. Scalable test problem with n "segments".

relaxed. As a test problem for our experiments, we chose a simple single-agent problem that was easily scalable and had optimal policies whose meaning was intuitively clear. The problem is shown in Figure 4. It is composed of n two-state segments and a sink state. Thus, the problem consists of $2n + 1$ states, which are numbered as shown in the figure. There are $n + 1$ actions, numbered from a_0 to a_n. Action a_0 is a noop that does not require any resources and can be interpreted as doing nothing. In each state $s_i,\ i \in [1, n]$ (upper row), the agent can choose to execute the noop a_0 and go to the next state s_{i+1} without getting any reward. Alternatively, the agent can execute action a_i that "matches" the current state, in which case it has an equal probability of either going to state s_{n+i} (lower row) or staying in s_i, receiving a reward of i in both cases. However, if the agent executes any other "non-matching" action in state $s_i,\ i \in [1, n]$, it goes to the sink state s_0 with certainty and incurs a large penalty of -100. The only available action in states $s_i, i \in [n+1, 2n-2]$ (lower row) is the noop a_o, which yields a reward of zero and takes the agent to s_{i-n+1}. There is one resource, and each action a_i requires i units of it.

For this problem it is easy to analytically compute the optimal policy, its value, and its resource requirements. In fact, this problem is equivalent to a knapsack problem where the value-to-cost ratio of all items is the same. In our experiments we varied the size of the problem (n) and the amount of the resource that was available to the agent (γ). The latter was measured as the fraction of the resource amount required by the optimal unconstrained policy. Figure 5a shows the value of the optimal policy as a function of γ and n.

We also compared our approach based on mixed integer programming to a very simple heuristic search method, which worked as follows. It first solved the unconstrained problem and then sequentially replaced some actions with the noop to reduce the cost of the policy. The actions to be replaced were chosen randomly. As can be seen from Figure 5b, which shows the values of the policies produced by the two methods, the greedy heuristic works very well for this test problem. This should not be surprising, given the analogy of this problem to knapsack, and the fact that all actions have the same reward-to-cost ratio. The fact that this elementary heuristic approach works well on some problems at a fraction of the time required for the MIP method suggests that exploring search-based approaches for this optimization problem might be worthwhile. Of course, this particular heuristic method turns out to have been well matched to this



(a) (b)

Fig. 5. (a) Value of optimal policies for the test problem; (b) Value of policies produced by the MIP approach (solid lines) and the greedy heuristic (dashed lines).

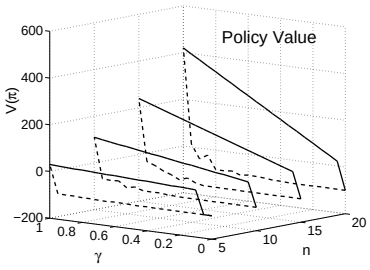


Fig. 6. Values of policies produced by the MIP approach (solid lines) and the greedy heuristic (dashed lines) for a slightly modified version of the problem in Figure 4. Now, the role of the noop and the other actions in the upper-row states $s_i, i \in [1, n]$ is reversed. The noop leads to the sink state s_0, incurring a penalty of -100 and the other "non-matching" actions lead to the next state s_{i+1} with no reward. For this problem, the greedy heuristic produces the worst possible policies for almost all values of γ.

problem only by good fortune. It can do very poorly on variations of this problem, as shown in Figure 6, where the heuristic almost always produces the worst possible policy. A systematic investigation of heuristic search-based methods is required before any claims can be made about the trade-offs and benefits of using such methods. This is one of the directions of our future work.

We also performed a preliminary evaluation of the scalability and the running time of our approach. Using CPLEX 8.1 as the MIP solver, on problems of size up to $n = 150$,[3] we were able to obtain optimal solutions for the problem in Figure 4 in under 30 seconds (in most cases, much faster). In some cases, when CPLEX was not able to prove optimality of the obtained solution in the alloted 30 seconds, it returned optimality gaps of several percent. With problems of larger size, we ran into memory issues, as our code at the time of writing does not yet exploit matrix sparsity, which results in highly inefficient use of memory space.

[3] For $n = 150$, the transition matrix has around 10^7 elements, the size of the policy search space is roughly 10^{650}, and there are about 45000 optimization variables in the MIP.

7 Conclusions

We have addressed the problem of optimal resource allocation and policy formulation for teams of cooperative agents in stochastic environments. We have analyzed the complexity of the problem and presented a solution algorithm, based on the reduction of the problem to a mixed integer program. This reduction allows us to make use of existing tools for efficiently solving MIPs. However, our methodology still suffers from the "curse of dimensionality", which plagues the standard "flat" MDP model. Recently, several methods have been proposed to alleviate this difficulty by exploiting the structure of the problem and working on parameterized or factored state and action spaces. Given the complexity of our problem, it appears that it could significantly benefit from these ideas. One of the directions of our future work is to try to address the complexity and scalability of our approach by utilizing parameterized MDP models.

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