Towards Tractable Local Closed World Reasoning for the Semantic Web*

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Abstract. Recently, the logics of minimal knowledge and negation as failure MKNF [11] was used to introduce hybrid MKNF knowledge bases [13], a powerful formalism for combining open and closed world reasoning for the Semantic Web. We present an extension based on a new three-valued framework including an alternating fixpoint, the well-founded MKNF model. This approach, the well-founded MKNF semantics, derives its name from the very close relation to the corresponding semantics known from logic programming. We show that the well-founded MKNF model is the least model among all (three-valued) MKNF models, thus soundly approximating also the two-valued MKNF models from [13]. Furthermore, its computation yields better complexity results (up to polynomial) than the original semantics where models usually have to be guessed.

1 Introduction

Joining the open-world semantics of DLs with the closed-world semantics featured by (nonmonotonic) logic programming (LP) is one of the major open research questions in Description Logics (DL) research. Indeed, adding rules, in LP style, on top of the DL-based ontology layer has been recognized as an important task for the success of the Semantic Web (cf. the Rule Interchange Format working group of the W3C[3]). Combining LP rules and DLs, however, is a non-trivial task since these two formalisms are based on different assumptions: the former is nonmonotonic, relying on the closed world assumption, while the latter is based on first-order logic under the open world assumption.

Several proposals have been made for dealing with knowledge bases (KBs) which contain DL and LP statements (see e.g. [2–4, 8, 13, 14]), but apart from [4], they all rely on the stable models semantics (SMS) for logic programs [6]. We claim that the well-founded semantics (WFS) [16], though being closely related to SMS (see e.g. [5]), is often the better choice. Indeed, in applications dealing with a large amount of information like the Semantic Web, the polynomial worst-case complexity of WFS is preferable

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[3] http://www.w3.org/2005/rules/

to the NP-hard SMS. Furthermore, the WFS is defined for all programs and allows to answer queries by consulting only the relevant part of a program whereas SMS is neither relevant nor always defined.

While the approach in [4] is based on a loose coupling between DL and LP, others are tightly integrated. The most advanced of these approaches currently appears to be that of hybrid MKNF knowledge bases [13] which is based on the logic of Minimal Knowledge and Negation as Failure (MKNF) [11]. Its advantage lies in a seamless integration of DL and LP within one logical framework while retaining decidability due to the restriction to DL-safe rules.

In this paper[4], we define a three-valued semantics for hybrid MKNF knowledge bases, and a well-founded semantics, restricted to nondisjunctive MKNF rules, whose only model is the least three-valued one wrt. derivable knowledge. It compares to the semantics of [13] as the WFS does to the SMS of LP, viz.:

- the well-founded semantics is a sound approximation of the semantics of [13];
- the computational complexity is strictly lower;
- the semantics retains the property of [13] of being faithful, but now wrt. the WFS, i.e. when the DL part is empty, it coincides with the WFS of LPs.

2 Preliminaries

MKNF notions. We start by recalling the syntax of MKNF formulas from [13]. A *first-order atom* $P(t_1, \ldots, t_n)$ is an MKNF formula where P is a predicate and the t_i are first-order terms[5]. If φ is an MKNF formula then $\neg\varphi$, $\exists x : \varphi$, $\mathbf{K}\,\varphi$ and $\mathbf{not}\,\varphi$ are MKNF formulas and likewise $\varphi_1 \wedge \varphi_2$ for MKNF formulas φ_1, φ_2. The symbols $\vee$, $\subset$, $\equiv$, and $\forall$ are abbreviations for the usual boolean combinations of the previously introduced syntax. Substituting the free variables x_i in φ by terms t_i is denoted $\varphi[t_1/x_1, \ldots, t_n/x_n]$. Given a (first-order) formula φ, $\mathbf{K}\,\varphi$ is called a *modal* $\mathbf{K}$*-atom* and $\mathbf{not}\,\varphi$ a *modal* $\mathbf{not}$*-atom*. If a modal atom does not occur in scope of a modal operator in an MKNF formula then it is *strict*. An MKNF formula φ without any free variables is a *sentence* and *ground* if it does not contain variables at all. It is *modally closed* if all modal operators ($\mathbf{K}$ and $\mathbf{not}$) are applied in φ only to sentences and *positive* if it does not contain the operator $\mathbf{not}$; φ is *subjective* if all first-order atoms of φ occur within the scope of a modal operator and *objective* if there are no modal operators at all in φ; φ is *flat* if it is subjective and all occurrences of modal atoms in φ are strict.

Apart from the constants occurring in the formulas, the signature contains a countably infinite supply of constants not occurring in the formulas. The Herbrand Universe of such a signature is also denoted $\triangle$. The signature contains the equality predicate $\approx$ which is interpreted as an equivalence relation on $\triangle$. An *MKNF structure* is a triple (I, M, N) where I is an Herbrand first-order interpretation over $\triangle$ and M and N are nonempty sets of Herbrand first-order interpretations over $\triangle$. MKNF structures (I, M, N) define satisfiability of MKNF sentences as follows:

[4] Preliminary work on this subject was presented in [10].

[5] We consider function-free first-order logic, so terms are either constants or variables.

$(I, M, N) \models p(t_1, \ldots, t_n)$ iff $p(t_1, \ldots, t_n) \in I$
$(I, M, N) \models \neg\varphi$ iff $(I, M, N) \not\models \varphi$
$(I, M, N) \models \varphi_1 \wedge \varphi_2$ iff $(I, M, N) \models \varphi_1$ and $(I, M, N) \models \varphi_2$
$(I, M, N) \models \exists x : \varphi$ iff $(I, M, N) \models \varphi[\alpha/x]$ for some $\alpha \in \triangle$
$(I, M, N) \models \mathbf{K}\, \varphi$ iff $(J, M, N) \models \varphi$ for all $J \in M$
$(I, M, N) \models \mathbf{not}\, \varphi$ iff $(J, M, N) \not\models \varphi$ for some $J \in N$

An *MKNF interpretation* M is a nonempty set of Herbrand first-order interpretations over[6] $\triangle$ and *models* a closed MKNF formula φ, i.e. $M \models \varphi$, if $(I, M, M) \models \varphi$ for each $I \in M$. An MKNF interpretation M is an *MKNF model* of a closed MKNF formula φ if (1) M models φ and (2) for each MKNF interpretation M' such that $M' \supset M$ we have $(I', M', M) \not\models \varphi$ for some $I' \in M'$.

Hybrid MKNF Knowledge Bases. Quoting from [13], the approach of hybrid MKNF knowledge bases is applicable to any first-order fragment $\mathcal{DL}$ satisfying these conditions: (i) each knowledge base $\mathcal{O} \in \mathcal{DL}$ can be translated into a formula $\pi(\mathcal{O})$ of function-free first-order logic with equality, (ii) it supports *A-Boxes*-assertions of the form $P(a_1, \ldots, a_n)$ for P a predicate and a_i constants of $\mathcal{DL}$ and (iii) satisfiability checking and instance checking (i.e. checking entailment of the form $\mathcal{O} \models P(a_1, \ldots, a_n)$ is decidable[7].

We recall MKNF rules and hybrid MKNF knowledge bases from [13]. For the rational behind these and the following notions we also refer to [12].

Definition 2.1. *Let $\mathcal{O}$ be a description logics knowledge base. A first-order function-free atom $P(t_1, \ldots, t_n)$ over Σ such that P is $\approx$ or it occurs in $\mathcal{O}$ is called a* DL-atom*; all other atoms are called* non-DL-atoms. *An MKNF rule r has the following form where H_i, A_i, and B_i are first-order function free atoms:*

$$\mathbf{K}\, H_1 \vee \ldots \vee \mathbf{K}\, H_l \leftarrow \mathbf{K}\, A_1, \ldots, \mathbf{K}\, A_n, \mathbf{not}\, B_1, \ldots, \mathbf{not}\, B_m \tag{1}$$

The sets $\{\mathbf{K}\, H_i\}$, $\{\mathbf{K}\, A_i\}$, and $\{\mathbf{not}\, B_i\}$ are called the rule head*, the* positive body*, and the* negative body*, respectively. A rule is* nondisjunctive *if $l = 1$; r is* positive *if $m = 0$; r is a* fact *if $n = m = 0$. A* program *is a finite set of MKNF rules. A* hybrid MKNF knowledge base *$\mathcal{K}$ is a pair $(\mathcal{O}, \mathcal{P})$ and $\mathcal{K}$ is* nondisjunctive *if all rules in $\mathcal{P}$ are nondisjunctive.*

The semantics of an MKNF knowledge base is obtained by translating it into an MKNF formula ([13]).

Definition 2.2. *Let $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ be a hybrid MKNF knowledge base. We extend π to r, $\mathcal{P}$, and $\mathcal{K}$ as follows, where x is the vector of the free variables of r.*

$$\pi(r) = \forall x : (\mathbf{K}\, H_1 \vee \ldots \vee \mathbf{K}\, H_l \subset \mathbf{K}\, A_1, \ldots, \mathbf{K}\, A_n, \mathbf{not}\, B_1, \ldots, \mathbf{not}\, B_m)$$

$$\pi(\mathcal{P}) = \bigwedge_{r \in \mathcal{P}} \pi(r) \qquad \pi(\mathcal{K}) = \mathbf{K}\, \pi(\mathcal{O}) \wedge \pi(\mathcal{P})$$

[6] Due to the domain set $\triangle$, the considered interpretations are in general infinite.
[7] For more details on DL notation we refer to [1].

An MKNF rule r is *DL-safe* if every variable in r occurs in at least one non-DL-atom $\mathbf{K}\, B$ occurring in the body of r. A hybrid MKNF knowledge base $\mathcal{K}$ is *DL-safe* if all its rules are DL-safe. Given a hybrid MKNF knowledge base $\mathcal{K} = (\mathcal{O}, \mathcal{P})$, the *ground instantiation of* $\mathcal{K}$ is the KB $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ where $\mathcal{P}_G$ is obtained by replacing in each rule of $\mathcal{P}$ all variables with constants from $\mathcal{K}$ in all possible ways. It was shown in [12], for a DL-safe hybrid KB $\mathcal{K}$ and a ground MKNF formula ψ, that $\mathcal{K} \models \psi$ if and only if $\mathcal{K}_G \models \psi$.

3 Three-valued MKNF Semantics

Satisfiability as defined before allows modal atoms only to be either true or false in a given MKNF structure. We extend the framework with a third truth value $\mathbf{u}$, denoting undefined, to be assigned to modal atoms only, while first-order atoms remain two-valued due to being interpreted solely in one first-order interpretation. So MKNF sentences are evaluated in MKNF structures with respect to the set $\{\mathbf{t}, \mathbf{u}, \mathbf{f}\}$ of truth values with the order $\mathbf{f} < \mathbf{u} < \mathbf{t}$, and where the operator $\max$ (resp. $\min$) chooses the greatest (resp. least) element with respect to this ordering:

- $(I, M, N)(p(t_1, \ldots, t_n)) = \begin{cases} \mathbf{t} \text{ iff } p(t_1, \ldots, t_n) \in I \\ \mathbf{f} \text{ iff } p(t_1, \ldots, t_n) \notin I \end{cases}$
- $(I, M, N)(\neg\varphi) = \begin{cases} \mathbf{t} \text{ iff } (I, M, N)(\varphi) = \mathbf{f} \\ \mathbf{u} \text{ iff } (I, M, N)(\varphi) = \mathbf{u} \\ \mathbf{f} \text{ iff } (I, M, N)(\varphi) = \mathbf{t} \end{cases}$
- $(I, M, N)(\varphi_1 \wedge \varphi_2) = \min\{(I, M, N)(\varphi_1), (I, M, N)(\varphi_2)\}$
- $(I, M, N)(\varphi_1 \supset \varphi_2) = \mathbf{t}$ iff $(I, M, N)(\varphi_2) \geq (I, M, N)(\varphi_1)$ and $\mathbf{f}$ otherwise
- $(I, M, N)(\exists x : \varphi) = \max\{(I, M, N)(\varphi[\alpha/x]) \mid \alpha \in \triangle\}$
- $(I, M, N)(\mathbf{K}\, \varphi) = \begin{cases} \mathbf{t} \text{ iff } (J, M, N)(\varphi) = \mathbf{t} \text{ for all } J \in M \\ \mathbf{f} \text{ iff } (J, M, N)(\varphi) = \mathbf{f} \text{ for some } J \in N \\ \mathbf{u} \text{ otherwise} \end{cases}$
- $(I, M, N)(\mathbf{not}\, \varphi) = \begin{cases} \mathbf{t} \text{ iff } (J, M, N)(\varphi) = \mathbf{f} \text{ for some } J \in N \\ \mathbf{f} \text{ iff } (J, M, N)(\varphi) = \mathbf{t} \text{ for all } J \in M \\ \mathbf{u} \text{ otherwise} \end{cases}$

To avoid having modal atoms which are true and false at the same time we restrict MKNF structures to consistent ones.

Definition 3.1. *An MKNF structure (I, M, N) is called* consistent *if, for all MKNF formulas φ over some given signature, it is not the case that $(J, M, N)(\varphi) = \mathbf{t}$ for all $J \in M$, nor is it the case that $(J, M, N)(\varphi) = \mathbf{f}$ for some $J \in N$.*

First of all, this evaluation is not really a purely three-valued one since first-order atoms are evaluated like in the two-valued case. In fact, a pure description logic knowledge base is only two-valued and it can easily be seen that it is evaluated in exactly the same way as in the scheme presented in the previous section. This is desired in particular in the case when the knowledge base consists just of the DL part. The third truth value can thus only be rooted in the rules part of the knowledge base. So, the main difference to the previous two-valued scheme consists of two pieces:

1. Implications are no longer interpreted classically: $\mathbf{u} \leftarrow \mathbf{u}$ is true while the classical boolean correspondence is $\mathbf{u} \vee \neg \mathbf{u}$, respectively $\neg(\neg \mathbf{u} \wedge \mathbf{u})$, which is undefined. The reason for this change is that rules in this way can only be true or false, similarly to logic programming, even in the case of three-valued semantics.
2. While in the two-valued framework M is used solely for interpreting modal $\mathbf{K}$-atoms and N only for the evaluation of modal $\mathbf{not}$-atoms, the three-valued evaluation applies symmetrically both sets to each case.

The second point needs further explanations. In the two-valued scheme, $\mathbf{K}\,\varphi$ is true in a given MKNF structure (I, M, N) if it holds in all Herbrand interpretations occurring in M and false otherwise, and in case of $\mathbf{not}$ exactly the other way around wrt. N. However, the truth space is thus fully defined leaving no gap for undefined modal atoms. One could change the evaluation such that e.g. $\mathbf{K}\,\varphi$ is true in M if φ is true in all $J \in M$, false in M if φ is false in all $J \in M$, and undefined otherwise. Then $\mathbf{not}\,\varphi$ would only be true if it is false in all models in N and we no longer have a negation different from the classical one. Thus, we separate truth and falsity in the sense that whenever a modal atom $\mathbf{K}\,\varphi$ is not true in M then it is either false or undefined. The other set, N, then allows to obtain whether $\mathbf{K}\,\varphi$ is false, namely just in case $\mathbf{not}\,\varphi$ is true[8]. We only have to be careful regarding consistency: we do not want structures which evaluate modal atoms to true and false at the same time and thus also not that $\mathbf{K}\,\varphi$ and $\mathbf{not}\,\varphi$ are true with respect to the same MKNF structure. The last case might in fact occur in the two-valued evaluation but does not do any harm there since the explicit connection between $\mathbf{K}\,\varphi$ and $\mathbf{not}\,\varphi$ is not present in the evaluation, and these inconsistencies are afterwards inhibited in MKNF interpretations.

Obviously, MKNF interpretations are not suitable to represent three truth values. For this purpose, we introduce interpretation pairs.

Definition 3.2. *An* interpretation pair (M, N) *consists of two MKNF interpretations M, N with $N \subseteq M$, and* models *a closed MKNF formula φ, written $(M, N) \models \varphi$, if and only if $(I, M, N)(\varphi) = \mathbf{t}$ for each $I \in M$. We call (M, N) consistent if (I, M, N) is consistent for any $I \in M$ and φ* consistent *if there exists an interpretation pair modeling it.*

M contains all interpretations which model only truth while N models everything which is true or undefined. Evidently, just as in the two-valued case, anything not being modeled in N is false.

We now introduce a preference relation on pairs in a straightforward way.

Definition 3.3. *Given a closed MKNF formula φ, a (consistent) interpretation pair (M, N) is a* (three-valued) MKNF model *for φ if $(1) (I, M, N)(\varphi) = \mathbf{t}$ for all $I \in M$ and (2) for each MKNF interpretation M' with $M' \supset M$ we have $(I', M', N)(\varphi) = \mathbf{f}$ for some $I' \in M'$.*

The idea is, having fixed the evaluation in N, i.e. the modal $\mathbf{K}$-atoms which are false (and thus also the modal $\mathbf{not}$-atoms which are true), to maximize the set which

[8] This concurs with the idea that $\mathbf{not}$ is meant to represent $\neg\mathbf{K}$

evaluates modal $\mathbf{K}$-atoms to true, thus only incorporating all the minimally necessary knowledge into M. In this sense, we remain in a logic of minimal knowledge. As a side-effect, we also minimize the falsity of modal $\mathbf{not}$-atoms, which is justified by the relation of $\mathbf{K}$ and $\neg\mathbf{not}$. This feature is not contained in the MKNF semantics, but not necessary in the two-valued case anyway. We nevertheless obtain that any (two-valued) MKNF model M corresponds exactly to a three-valued one.

Proposition 3.1. *Given a closed MKNF formula φ, if M is an MKNF model of φ then (M, M) is a three-valued MKNF model of φ.*

Example 3.1. Consider the following knowledge base $\mathcal{K}$ containing just two rules.

$$\mathbf{K}\, p \leftarrow \mathbf{not}\, q$$
$$\mathbf{K}\, q \leftarrow \mathbf{not}\, p$$

The MKNF models of $\mathcal{K}$ are $\{\{p\}, \{p, q\}\}$ and $\{\{q\}, \{p, q\}\}$, i.e. $\mathbf{K}\, p$ and $\mathbf{not}\, q$ are true in the first model, and $\mathbf{K}\, q$ and $\mathbf{not}\, p$ are true in the second one.

We thus obtain two three-valued MKNF models: $(\{\{p\}, \{p, q\}\}, \{\{p\}, \{p, q\}\})$ and $(\{\{q\}, \{p, q\}\}, \{\{q\}, \{p, q\}\})$. Besides that, any interpretation pair which maps $\mathbf{K}\, p$, $\mathbf{K}\, q$, $\mathbf{not}\, p$ and $\mathbf{not}\, q$ to undefined is also a three-valued model. Among those, only $(\{\emptyset, \{p\}, \{q\}, \{p, q\}\}, \{\{p, q\}\})$ is an MKNF model while e.g. $(\{\emptyset, \{p, q\}\}, \{\{p, q\}\})$ is not. Finally, the pair which maps both, p and q, to true is a model but not MKNF either since $(\{\{p, q\}\}, \{\{p, q\}\})$ is also dominated by $(\{\emptyset, \{p\}, \{q\}, \{p, q\}\}, \{\{p, q\}\})$

There is one alternative idea for defining three-valued structures. We can represent a first-order interpretation by the set of all atoms which are true and the set of all negated atoms which are false. Thus, in the previous example we would obtain sets consisting of p, q, $\neg p$ and $\neg q$ where e.g. $\{p, \neg q\}$ instead of $\{p\}$ represents that p is true and q is false. This results in an MKNF model $\{\{p, \neg q\}\}$ which represents the knowledge $\mathbf{K}\, p$ and $\mathbf{K}\, \neg q$, respectively $\mathbf{not}\, \neg p$ and $\mathbf{not}\, q$. Unfortunately, for the three-valued model we obtain a representation $(\{\{p, \neg q\}, \{p, q\}, \{\neg p, \neg q\}, \{\neg p, q\}\}, \{\{p, q, \neg p, \neg q\}\})$ leaving p and q undefined. This is not very useful since it forces us to include inconsistent interpretations into interpretation pairs to state e.g. that neither $\mathbf{K}\, \neg p$ nor $\mathbf{K}\, p$ hold.

4 Three-valued MKNF Models and Partitions

As shown in [12], since MKNF models are in general infinite, they can better be represented via a 1st-order formula whose models are exactly contained in the considered MKNF model. The idea is to provide a partition (T, F) of true and false modal atoms which uniquely defines φ. The 1st-order formula is then obtained from T as the objective knowledge contained in the modal atoms. We extend this idea to partial partitions where modal atoms which neither occur in T nor in F are supposed to be undefined. To obtain a specific partial partition we apply a technique known from LP: stable models ([6]) for normal logic programs correspond one-to-one to MKNF models of programs of MKNF rules (see [11]) and the well-founded model ([16]) for normal logic programs can be computed by an alternating fixpoint of the operator used to define stable models ([15]).

We proceed similarly: we define an operator providing a stable condition for nondisjunctive hybrid MKNF knowledge bases and use it to obtain an alternating fixpoint, the well-founded semantics. We thus start by adapting some notions from [12] formalizing partitions and related concepts.

Definition 4.1. *Let σ be a flat modally closed MKNF formula. The* set of $\mathbf{K}$-atoms of *σ, written $\mathsf{KA}(\sigma)$, is the smallest set that contains (i) all $\mathbf{K}$-atoms occurring in σ, and (ii) a modal atom $\mathbf{K}\,\xi$ for each modal atom $\mathbf{not}\,\xi$ occurring in σ.*

For a subset S of $\mathsf{KA}(\sigma)$, the objective knowledge *of S is the formula $\mathsf{ob}_S = \bigcup_{\mathbf{K}\,\xi \in S} \xi$. A* (partial) partition *(T, F) of $\mathsf{KA}(\sigma)$ is* consistent *if $\mathsf{ob}_T \not\models \xi$ for each $\mathbf{K}\,\xi \in F$.*

We now connect interpretation pairs and partial partitions of modal $\mathbf{K}$-atoms similarly to the way it was done in [12].

Definition 4.2. *We say that a partial partition (T, F) of $\mathsf{KA}(\sigma)$ is* induced by *a consistent interpretation pair (M, N) if (1) whenever $\mathbf{K}\,\xi \in T$ then $M \models \mathbf{K}\,\xi$ and $N \models \mathbf{K}\,\xi$, (2) whenever $\mathbf{K}\,\xi \in F$ then $N \not\models \mathbf{K}\,\xi$, and (3) whenever $\mathbf{K}\,\xi \in T$ or $\mathbf{K}\,\xi \in F$ then it is not the case that $M \not\models \mathbf{K}\,\xi$ and $N \models \mathbf{K}\,\xi$.*

The only case not dealt with in this definition is the one where M models $\mathbf{K}\,\xi$ and N does not model $\mathbf{K}\,\xi$, i.e. N modeling $\mathbf{not}\,\xi$. But this cannot occur since interpretation pairs are restricted to consistent ones.

Based on this relation we can show that the objective knowledge derived from the partition which is induced by a three-valued MKNF model yields again that model.

Proposition 4.1. *Let σ be a flat modally closed MKNF formula, (M, N) an MKNF model of σ and (T, F) a partition of $\mathsf{KA}(\sigma)$ induced by (M, N). Then (M, N) is equal to the interpretation pair (M', N') where $M' = \{I \mid I \models \mathsf{ob}_T\}$ and $N' = \{I \mid I \models \mathsf{ob}_{\mathsf{KA}(\sigma)\setminus F}\}$.*

Proof. Let I be an interpretation in M. Since (M, N) induces the partition (T, F) for each $\mathbf{K}\,\xi \in T$ we have $M \models \mathbf{K}\,\xi$ and thus $I \models \xi$. Hence, $I \models \mathsf{ob}_T$ which shows $M \subseteq M'$. Likewise, for each $\mathbf{K}\,\xi \notin F$ we have $N \models \mathbf{K}\,\xi$ and so for each $I \in N$ it holds that $I \models \xi$. Then $I \models \mathsf{ob}_{\mathsf{KA}(\sigma)\setminus F}$ which also shows $N \subseteq N'$.

Conversely, consider at first any I' in N'. We know for all $I' \in N'$ that $I' \models \mathsf{ob}_{\mathsf{KA}(\sigma)\setminus F}$, i.e. $I' \models \bigcup_{\mathbf{K}\,\xi \in \mathsf{KA}(\sigma)\setminus F} \xi$. Thus $I' \models \xi$ holds for all $\mathbf{K}\,\xi$ occurring in T and for all $\mathbf{K}\,\xi$ that neither occur in T nor in F. Since the partition was induced by (M, N) we obtain in both cases that $N \models \mathbf{K}\,\xi$, i.e. for all $I \in N$ we have $I \models \xi$ for all $\mathbf{K}\,\xi \in \mathsf{KA}(\sigma) \setminus F$. We conclude that $N = N'$. For showing that also $M = M'$ we assume that $M' \setminus M$ is not empty but contains an interpretation I'. Then, for each $K\xi \in T$ we obtain $(I', M', N)(\mathbf{K}\,\xi) = \mathbf{t}$ just as we have $(I, M, N)(\mathbf{K}\,\xi) = \mathbf{t}$ for all $I \in M$. Likewise, for each $K\xi \in T$ we have $(I', M', N)(\mathbf{not}\,\xi) = \mathbf{f}$ for any $I' \in M'$ and $(I, M, N)(\mathbf{not}\,\xi) = \mathbf{f}$ for any $I \in M$. We also know for each $K\xi \in F$ that $(I', M', N)(\mathbf{not}\,\xi) = \mathbf{t}$ and $(I, M, N)(\mathbf{not}\,\xi) = \mathbf{t}$, and $(I', M', N)(\mathbf{K}\,\xi) = \mathbf{f}$ and $(I, M, N)(\mathbf{K}\,\xi) = \mathbf{f}$ for any I, I' since N remains the same. For the same reason, and since augmenting M does not alter the undefinedness of a modal atom, all modal atoms which are undefined in (I, M, N) are also undefined in (I, M', N) The truth value of a flat σ in a structure (I', M', N) for some $I' \in M'$ is completely defined by the truth values of the modal atoms and since they are all identical to the ones in (I, M, N) for all $I \in M$ we have that $(I', M', N)(\sigma) = \mathbf{t}$. This contradicts the assumption that M is a three-valued MKNF model of σ.

We will use this result later on to show that the partition obtained from the alternating fixpoint yields in fact a three-valued MKNF model. For that, we adapt some more notions like in [13].

Consider a hybrid MKNF knowledge base $\mathcal{K} = (\mathcal{O}, \mathcal{P})$. Note that $\mathbf{K}\,\pi(\mathcal{O})$ occurs in $\mathsf{KA}(\sigma)$ and must be true in any model of $\mathcal{K}$. The set of the remaining modal $\mathbf{K}$-atoms is denoted $\mathsf{KA}(\mathcal{K}) = \mathsf{KA}(\sigma) \setminus \{\mathbf{K}\,\pi(\mathcal{O})\}$. Furthermore, for a set of modal atoms S, S_{DL} is the subset of DL-atoms of S, and $\widehat{S} = \{\xi \mid \mathbf{K}\,\xi \in S\}$. These changes allow to rewrite the objective knowledge in the following way where S is a subset of $\mathsf{KA}(\mathcal{K})$: $\mathsf{ob}_{\mathcal{K},S} = \mathcal{O} \cup \bigcup_{\mathbf{K}\,\xi \in S} \xi$.

We now adapt the monotonic operator $T_\mathcal{K}$ from [13] which allows to draw conclusions from positive hybrid MKNF knowledge bases.

Definition 4.3. *For $\mathcal{K}$ a positive nondisjunctive DL-safe hybrid MKNF knowledge base, $R_\mathcal{K}$, $D_\mathcal{K}$, and $T_\mathcal{K}$ are defined on the subsets of $\mathsf{KA}(\mathcal{K})$ as follows:*

$$\begin{aligned}
R_\mathcal{K}(S) = {} & S \cup \{\mathbf{K}\,H \mid \mathcal{K} \textit{ contains a rule of the form (1) such that } \mathbf{K}\,A_i \in S \\
& \quad \textit{for each } 1 \le i \le n\} \\
D_\mathcal{K}(S) = {} & \{\mathbf{K}\,\xi \mid \mathbf{K}\,\xi \in \mathsf{KA}(\mathcal{K}) \textit{ and } \mathcal{O} \cup \widehat{S}_{DL} \models \xi\} \cup \\
& \{\mathbf{K}\,Q(b_1, \ldots, b_n) \mid \mathbf{K}\,Q(a_1, \ldots, a_n) \in S \setminus S_{DL}, \mathbf{K}\,Q(b_1, \ldots, b_n) \in \mathsf{KA}(\mathcal{K}), \\
& \quad \textit{and } \mathcal{O} \cup \widehat{S}_{DL} \models a_i \approx b_i \textit{ for } 1 \le i \le n\} \\
T_\mathcal{K}(S) = {} & R_\mathcal{K}(S) \cup D_\mathcal{K}(S)
\end{aligned}$$

The difference to the operator $D_\mathcal{K}$ in [13] is that given e.g. only $a \approx b$ and $\mathbf{K}\,Q(a)$ we do not derive $\mathbf{K}\,Q(b)$ explicitly but only as a consequence of $\mathsf{ob}_{\mathcal{K},P}$.

A transformation for nondisjunctive hybrid MKNF knowledge is defined turning them into positive ones, thus allowing the application of the operator $T_\mathcal{K}$.

Definition 4.4. *Let $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ be a ground nondisjunctive DL-safe hybrid MKNF knowledge base and $S \subseteq \mathsf{KA}(\mathcal{K})$. The* MKNF transform *$\mathcal{K}_G/S = (\mathcal{O}, \mathcal{P}_G/S)$ is obtained by $\mathcal{P}_G/S$ containing all rules $\mathbf{K}\,H \leftarrow \mathbf{K}\,A_1, \ldots, \mathbf{K}\,A_n$ for which there exists a rule $\mathbf{K}\,H \leftarrow \mathbf{K}\,A_1, \ldots, \mathbf{K}\,A_n, \mathbf{not}\,B_1, \ldots, \mathbf{not}\,B_m$ in $\mathcal{P}_G$ with $\mathbf{K}\,B_j \notin S$ for all $1 \le j \le m$.*

Now an antitonic operator can be defined using the fixpoint of $T_\mathcal{K}$.

Definition 4.5. *Let $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ be a nondisjunctive DL-safe hybrid MKNF knowledge base and $S \subseteq \mathsf{KA}(\mathcal{K})$. We define:*

$$\Gamma_\mathcal{K}(S) = T_{\mathcal{K}_G/S} \uparrow \omega$$

Applying $\Gamma_\mathcal{K}(S)$ twice is a monotonic operation yielding a least fixpoint by the Knaster-Tarski theorem (and dually a greatest one) and can be iterated as follows: $\Gamma^2_\mathcal{K} \uparrow 0 = \emptyset$, $\Gamma^2_\mathcal{K} \uparrow (n+1) = \Gamma^2_\mathcal{K}(\Gamma^2_\mathcal{K} \uparrow n)$, and $\Gamma^2_\mathcal{K} \uparrow \omega = \bigcup \Gamma^2_\mathcal{K} \uparrow i$, and dually $\Gamma^2_\mathcal{K} \downarrow 0 = \mathsf{KA}(\mathcal{K})$, $\Gamma^2_\mathcal{K} \downarrow (n+1) = \Gamma^2_\mathcal{K}(\Gamma^2_\mathcal{K} \downarrow n)$, and $\Gamma^2_\mathcal{K} \downarrow \omega = \bigcap \Gamma^2_\mathcal{K} \downarrow i$.

These two fixpoints define the well-founded partition.

Definition 4.6. *Let $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ be a nondisjunctive DL-safe hybrid MKNF knowledge base and let $\mathbf{P}_\mathcal{K}, \mathbf{N}_\mathcal{K} \subseteq \mathsf{KA}(\mathcal{K})$ with $\mathbf{P}_\mathcal{K}$ being the least fixpoint of $\Gamma^2_\mathcal{K}$ and $\mathbf{N}_\mathcal{K}$ the greatest fixpoint. Then $(P_W, N_W) = (\mathbf{P}_\mathcal{K} \cup \{\mathbf{K}\,\pi(\mathcal{O})\}, \mathsf{KA}(\mathcal{K}) \setminus \mathbf{N}_\mathcal{K})$ is the* well-founded partition *of $\mathcal{K}$.*

It can be shown that the well-founded partition is consistent and that the least fixpoint can be computed directly from the greatest one and vice-versa similarly to the alternating fixpoint of normal logic programs [15].

Proposition 4.2. *Let $\mathcal{K}$ be a nondisjunctive DL-safe hybrid MKNF knowledge base. Then $\mathbf{P}_\mathcal{K} = \Gamma_\mathcal{K}(\mathbf{N}_\mathcal{K})$ and $\mathbf{N}_\mathcal{K} = \Gamma_\mathcal{K}(\mathbf{P}_\mathcal{K})$.*

Example 4.1. Let us consider the following hybrid MKNF knowledge base

$$\mathit{NaturalDeath} \sqsubseteq \mathit{Pay} \qquad \mathit{Suicide} \sqsubseteq \neg \mathit{Pay}$$

$$\begin{aligned}
\mathbf{K}\,\mathit{Pay}(x) &\leftarrow \mathbf{K}\,\mathit{murdered}(x), \mathbf{K}\,\mathit{benefits}(y,x), \mathbf{not}\,\mathit{responsible}(y,x)\\
\mathbf{K}\,\mathit{Suicide}(x) &\leftarrow \mathbf{not}\,\mathit{NaturalDeath}(x), \mathbf{not}\,\mathit{murdered}(x)\\
\mathbf{K}\,\mathit{murdered}(x) &\leftarrow \mathbf{not}\,\mathit{NaturalDeath}(x), \mathbf{not}\,\mathit{Suicide}(x)
\end{aligned}$$

based on which a life insurance company decides whether to pay or not the insurance. Additionally, we know that Mr. Jones who owned a life insurance was found death in his living room, a revolver on the ground. Then $\neg \mathit{NaturalDeath}(\mathit{jones})$ and the last two rules offer us a choice between commitment of suicide or murder. While there are two MKNF models in such a scenario, one concluding for payment and the other one not, the three-valued framework allows to assign $\mathbf{u}$ to both so that we delay this decision until the evidence is evaluated. Assume that the police investigation reveals that the known criminal Max is responsible for the murder, though not being detectable, so we cannot conclude $\mathit{Suicide}(\mathit{jones})$ while $\mathbf{K}\,\mathit{responsible}(\mathit{max},\mathit{jones})$ and $\mathbf{K}\,\mathit{murdered}(\mathit{jones})$ hold. Unfortunately (for the insurance company), the person benefitting from the insurance is the nephew Thomas who many years ago left the country, i.e. $\mathbf{K}\,\mathit{benefits}(\mathit{thomas},\mathit{jones})$. Computing the well-founded partition yields thus $\mathbf{K}\,\mathit{Pay}(\mathit{jones})$, so the company has to contact the nephew. However, being not satisfied with the payment, they also hire a private detective who finds out that Max is Thomas, having altered his personality long ago, i.e. we have $\mathit{thomas} \approx \mathit{max}$ in the hybrid KB. Due to $D_\mathcal{K}$ and grounding we now obtain a well-founded partition which contains $\mathbf{K}\,\mathit{responsible}(\mathit{thomas},\mathit{jones})$ and $\mathbf{K}\,\mathit{benefits}(\mathit{max},\mathit{jones})$ being true and the insurance is not paid any longer.

Further examples, and an involved discussion of the importance of hybrid MKNF knowledge bases for modeling knowledge in the semantic web can be found in [9]. Apart from that, [7] provides arguments for the usefullness of epistemic reasoning the way it is done in MKNF logics.

We can also show that the well-founded partition yields a three-valued model.

Theorem 4.1. *Let $\mathcal{K}$ be a consistent nondisjunctive DL-safe hybrid MKNF KB and $(\mathbf{P}_\mathcal{K} \cup \{\mathbf{K}\,\pi(\mathcal{O})\}, \mathsf{KA}(\mathcal{K}) \setminus \mathbf{N}_\mathcal{K})$ be the well-founded partition of $\mathcal{K}$. Then $(I_P, I_N) \models \pi(\mathcal{K})$ where $I_P = \{I \mid I \models \mathsf{ob}_{\mathcal{K},\mathbf{P}_\mathcal{K}}\}$ and $I_N = \{I \mid I \models \mathsf{ob}_{\mathbf{N}_\mathcal{K}}\}$.*

Note that the DL-part of the knowledge base is not used for forming I_N. Otherwise I_N could be inconsistent since our approach might contain an undefined modal atom $\mathbf{K}\,\varphi$ even though φ is first-order false in the DL part. We are aware that this will need to be improved in further investigations. However, the deficiency is not severe in terms of the contribution of this work, since I_N is not used to evaluate the DL-part.

Now we combine this result with our previously proven proposition and obtain that the well-founded partition gives us in fact a three-valued MKNF model.

Theorem 4.2. *Let $\mathcal{K}$ be a consistent nondisjunctive DL-safe hybrid MKNF KB and $(\mathbf{P}_\mathcal{K} \cup \{\mathbf{K}\,\pi(\mathcal{O})\}, \mathsf{KA}(\mathcal{K}) \setminus \mathbf{N}_\mathcal{K})$ be the well-founded partition of $\mathcal{K}$. Then (I_P, I_N) where $I_P = \{I \mid I \models \mathsf{ob}_{\mathcal{K},\mathbf{P}_\mathcal{K}}\}$ and $I_N = \{I \mid I \models \mathsf{ob}_{\mathbf{N}_\mathcal{K}}\}$ is an MKNF model – the* well-founded MKNF model.

We begin by showing that the partition induced by a three-valued MKNF model is a fixpoint of $\Gamma^2_\mathcal{K}$.

Lemma 4.1. *Let $\mathcal{K}$ be a consistent nondisjunctive DL-safe hybrid MKNF KB and let (T, F) be the partition induced by an MKNF model (M, N) of $\mathcal{K}$. Then T and F are fixpoints of $\Gamma^2_\mathcal{K}$.*

Proof. (sketch) We show the argument for T; F follows dually. The set T contains all modal $\mathbf{K}$-atoms from $\mathsf{KA}(\mathcal{K})$ which are true in the MKNF model. We know that $\Gamma^2_\mathcal{K}$ is monotonic, i.e. $T \subseteq \Gamma^2_\mathcal{K}(T)$. Assume that $T \subset \Gamma^2_\mathcal{K}(T)$ so there are new consequences when applying $\Gamma^2_\mathcal{K}$ to T. Basically these modal atoms are justified either by $R_\mathcal{K}$ or by $D_\mathcal{K}$ or by some other new consequence of the application of $\Gamma^2_\mathcal{K}$. It is well known for this kind of rule languages (from logic programming theory) that consequences which depend on each other are established in a well-defined manner meaning that there are no cyclic dependencies (otherwise these modal atoms would remain undefined). We thus directly consider only basic new consequences which depend on no other new modal atom and restrict to the first two cases starting with $R_\mathcal{K}$.

If $\mathbf{K}\,H$ is a new consequence by means of $R_\mathcal{K}$ then it is because some rule of the form $\mathbf{K}\,H \leftarrow \mathbf{K}\,A_1, \ldots, \mathbf{K}\,A_n$ occurs in $\mathcal{K}_G/T$. Since $\mathbf{K}\,H$ does not depend on any other new consequence we already know that all $\mathbf{K}\,A_i$ are present in T. But $\mathbf{K}\,H$ does not occur in T, so there must be a modal atom $\mathbf{not}\,B_j$ in the corresponding rule such that $\mathbf{K}\,B_j \notin T$; otherwise (M, N) would be no model. This $\mathbf{K}\,B_j$ cannot occur in F either, otherwise (M, N) would again be no model of $\mathcal{K}$. Then $\mathbf{K}\,B_j$ is undefined but it is not possible to derive the falsity of $\mathbf{K}\,B_j$ and from that further conclusions in one step of $\Gamma^2_\mathcal{K}$.

Alternatively, $\mathbf{K}\,H$ is a consequence of $D_\mathcal{K}$. But then, according to the definition of that operator, any new consequence depends on some prior consequences or on the DL knowledge base itself. It is easy to see that thus new consequences derived from $D_\mathcal{K}$ depend on a new consequence introduced from the rules part; otherwise (M, N) would not be a model of $\mathcal{K}$.

From that we immediately obtain that the well-founded MKNF model is the least MKNF model wrt. derivable knowledge.

Theorem 4.3. *Let $\mathcal{K}$ be a consistent nondisjunctive DL-safe hybrid MKNF KB. Among all three-valued MKNF models, the well-founded MKNF model is the least wrt. derivable knowledge from $\mathcal{K}$.*

Proof. (sketch) We have shown that any three-valued MKNF model induces a partition which yields the MKNF model again (via the objective knowledge). Since this partition (T, F) consists of two fixpoints of $\Gamma_{\mathcal{K}}^2$ and we know that the well-founded partition (P_W, N_W) contains the least fixpoint (minimally necessary true knowledge) and the greatest fixpoint (minimally necessary false knowledge) we conclude that $P_W \subseteq T$ and $N_W \subseteq F$. It is straightforward to see that an MKNF model containing more true (and false) modal atoms allows for a greater set of logical consequences.

Thus, the well-founded partition can also be used in the algorithms presented in [13] for computing a subset of that knowledge which holds in all partitions corresponding to a two-valued MKNF model.

One of the open questions in [13] was that MKNF models are not compatible with the well-founded model for logic programs. Our approach, regarding knowledge bases just consisting of rules, does coincide with the well-founded model for the corresponding (normal) logic program.

Finally the following theorem is obtained straightforwardly from the data complexity results for positive nondisjunctive MKNF knowledge bases in [13] where data complexity is measured in terms of A-Box assertions and rule facts.

Theorem 4.4. *Let $\mathcal{K}$ be a nondisjunctive DL-safe hybrid MKNF KB. Assuming that entailment of ground DL-atoms in $\mathcal{DL}$ is decidable with data complexity $\mathcal{C}$ the data complexity of computing the well-founded partition is in* $\mathrm{P}^{\mathcal{C}}$.

For comparison, the data complexity for reasoning with MKNF models in nondisjunctive programs is shown to be $\mathcal{E}^{\mathrm{P}^{\mathcal{C}}}$ where $\mathcal{E} = \mathrm{NP}$ if $\mathcal{C} \subseteq \mathrm{NP}$, and $\mathcal{E} = \mathcal{C}$ otherwise. Thus, computing the well-founded partition ends up in a strictly smaller complexity class than deriving the MKNF models. In fact, in case the description logic fragment is tractable,[9] we end up with a formalism whose model is computed with a data complexity of P.

5 Conclusions and Future Work

We have continued the work on hybrid MKNF knowledge bases providing an alternating fixpoint restricted to nondisjunctive rules within a three-valued extension of the MKNF logic. We basically achieve better complexity results by having only one model which is semantically weaker than any MKNF model defined in [13], but bottom-up computable. The well-founded model is not only a sound approximation of any three-valued MKNF model but a partition of modal atoms which can seemlessly be integrated in the reasoning algorithms presented for MKNF models in [13] thus reducing the difficulty of guessing the 'right' model. Future work shall include the extension to disjunctive rules, handling of paraconsistency, and a study on top-down querying procedures.

[9] See e.g. the W3C member submission on tractable fragments of OWL 1.1 at http://www.w3.org/Submission/owl11-tractable/.

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