Local Closed-World Reasoning with Description Logics under the Well-founded Semantics

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Abstract

An important question for the upcoming Semantic Web is how to best combine open-world ontology languages, such as the OWL-based ones, with closed-world rules paradigms. One of the most mature proposals for this combination is known as hybrid MKNF knowledge bases [41], which is based on an adaptation of the Stable Model Semantics to knowledge bases consisting of ontology axioms and rules. In this paper, we propose a well-founded semantics for such knowledge bases which promises to provide better efficiency of reasoning and is compatible both with the OWL-based semantics and the traditional Well-Founded Semantics for logic programs. Moreover, our proposal permits the detection of inconsistencies possibly occuring in tightly integrated ontology axioms and rules with only little additional effort. We also identify tractable fragments of the resulting language.

Key words: Knowledge Representation, Description Logics and Ontologies, Non-monotonic Reasoning, Logic Programming, Semantic Web

1. Introduction

. The Semantic Web has recently become a major source of inspiration for Knowledge Representation and Reasoning (KR). The underlying idea of the Semantic Web is to use KR techniques to enhance data on the World Wide Web with knowledge bases, in order to make this data available for processing by intelligent systems. Semantic Web has become a mature field of research and a considerable industrial uptake for the use of Semantic Web technologies on and off the web can be observed. Semantic Web is a topic which is clearly here to stay.

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1.1. Open vs. Closed World Reasoning

The most prominent expressive KR approach employed in Semantic Web research is based on Description Logics [3, 22]. In particular, the Web Ontology Language OWL[3] is based on the description logic $\mathcal{SHOIN}(D)$ and a recommended standard by the World Wide Web Consortium (W3C) for modeling Semantic Web knowledge bases, commonly known as *ontologies*.

Description Logics (DLs), in turn, bear a first-order predicate logic semantics, and are thus monotonic and adhere to the Open-World Assumption (OWA). This means that (negative) conclusions drawn from a knowledge base must be based on information explicitly present in the knowledge base. As such, DLs differ from other KR formalisms studied e.g. in nonmonotonic reasoning, which are usually based on the Closed-World Assumption (CWA), and generally assume all non-provable expressions to be false.

The decision to base OWL on the OWA appears to be a natural one in light of the envisioned applications on the World Wide Web, where the absence of a piece of knowledge should not generally be taken as an indication of it being false. However, there are also natural application scenarios where the CWA, or at least the partial closure of the knowledge base, is the more natural choice. Such scenarios can occur e.g. if ontology-based reasoning is done in conjunction with data stored in a database, which is usually considered to be complete in the sense that an item which is not present in the database should be considered false. As a case in point [44] describes a large case study about matching patient records for clinical trials criteria containing up to millions of assertions. In that clinical domain, open world reasoning is needed in radiology and laboratory data, because, for example, unless a lab test asserts a negative finding no arbitrary assumptions about the results of the test can be made. E.g. we can only be certain that some patient does not have a specific kind of cancer if the corresponding test has a negative result. However, in pharmacy data, the closed world assumption can be used to infer that a patient is not on a medication if this is not asserted. The work of [44] applies only open-world reasoning but claims that the usage of closed world reasoning in pharmacy data would be highly desirable and that the combination of OWA and CWA is an open problem in their work.

It is thus apparent, and frequently being voiced by application developers, that it would be favorable to have *local* closed-world modeling as an additional feature for ontology-based systems, i.e. to have modelling capabilities which allow to interpret parts of the knowledge base under the CWA, and other parts under the OWA. Such modelling capabilities would enhance the usability of OWL for knowledge modellers considerably. A corresponding study for matchmaking using Semantic Web Services, for example, is given in [19]. Other natural scenarios such as the one above occur in the medical domain.

In fact, life sciences, including medicine, are a prominently studied application area for OWL, and in this area several large-scale ontologies have been

[3]See http://www.w3.org/2004/OWL/ and for the revision, called OWL 2, see [25].

developed which are in practical use, e.g. GALEN[4] and SNOMED.[5] Ontologies like these provide unified medical terminologies for the management and exchange of clinical information. The knowledge bases typically consist of information about anatomy, diseases, procedures, drugs, etc., and their applications reach from medical record management to diagnostics support. SNOMED is for example used in the application which matches patient records for clinical trials criteria described above.

As another example we can consider that such a medical knowledge base is used to decide whether a certain anaesthetic is applied before surgery, depending on whether the patient is allergic to it or not. In case of an emergency this information might not be available so we model it with the so-called default negation: unless we know explicitly about an allergy we assume that the patient is not allergic and we apply the anaesthetic. Other examples can be found if we were to model exceptions in anatomical terminology, e.g. the existence of persons whose heart is actually on the right-hand side. Exception modelling is not directly possible in OWL and this lack has led to several investigations into combinations of closed-world rules paradigms with DLs. However, this area of research must still be considered to be in its early stages.

1.2. Combining Rules and Ontologies

Combining rules and ontologies is a non-trivial task since a naive combination is undecidable ([27]). In fact, both formalisms differ substantially on how decidability is achieved. In case of ontologies decidability is achieved by certain syntactic restrictions on the available first-order predicates and the way these can be related. Rules do not have these restrictions but are limited in their applicability to the finitely many different objects in the knowledge base. An immediate effect of these differences is the availability of expressive features that are each only available for one of the two approaches. Rules permit to express non-treeshape-like relationships ([54])[6] such as "an uncle is the brother of one's father,", integrity constraints [47] to state e.g. that a certain piece of information is explicitly present in the database, and closed-world reasoning and exceptions as presented above. Ontologies on the contrary allow open-world reasoning and reasoning with unbounded or infinite domains and are thus well-suited to represent many types of incomplete information and schema knowledge. For example, in rule-based formalisms one typically cannot say that "every person has a father and a mother who are both persons" without listing all the parents explicitly. A combination of rules and ontologies is therefore not only of interest for current applications in the web but also as a highly sophisticated means of knowledge representation in general.

The proposed solutions on such a joint formalism differ substantially but we follow [43] and want the following criteria to be satisfied.

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[4]http://www.opengalen.org/

[5]http://www.ihtsdo.org/snomed-ct/

[6]The DL SROIQ [28] also provides role composition axioms, which can be used to address some, but by no means all use cases.

- Faithfulness: The integration between DLs and rules should preserve the semantics of both formalisms—that is, the semantics of a hybrid KB in which one component is empty should be the same as the semantics of the other component. In other words, the addition of rules to a DL should not change the semantics of the DL and vice versa.

- Tightness: Rules should not be layered on top of a DL or vice versa; rather, the integration between a DL and rules should be tight in the sense that both the DL and the rule component should be able to contribute to the consequences of the other component.

- Flexibility: The hybrid formalism should be flexible and allow one to view the same predicate under both open- and closed-world interpretation. This allows the rules to enrich a DL with nonmonotonic consequences, and a DL to enrich the rules with the capabilities of taxonomic reasoning.

- Decidability: To obtain a practically useful formalism that can be used in applications such as the Semantic Web, the hybrid formalism should be at least decidable, and preferably of low worst-case complexity.

1.3. Hybrid MKNF and Stable Models vs. Well-founded Semantics

As shown in [43], among the various proposals (e.g. [7, 11, 12, 14, 32, 43, 49, 50]) for combining rules and ontologies the only one satisfying all the four criteria is known as Hybrid MKNF knowledge bases [43], which draws on the logics of Minimal Knowledge and Negation as Failure (MKNF) [37]. An involved discussion of the importance of Hybrid MKNF knowledge bases for modeling knowledge in the Semantic Web can be found in [26], and [19] and [20] provide arguments for the usefulness of epistemic reasoning the way it is done in MKNF logics. The proposal by Motik and Rosati ([43]) seamlessly integrates arbitrary decidable description logics with essentially (disjunctive) logic programming rules allowing to reason over a combination of monotonic open world knowledge and nonmonotonic closed world knowledge within one framework.

Several reasoning algorithms are presented for the framework and one result shows that the data complexity of reasoning with Hybrid MKNF knowledge bases is in many cases not higher than reasoning in the corresponding fragment of logic programming. Thus, adding an ontology to rules does in general not increase the data complexity when compared to rules alone, but the very same cannot be said for adding rules to ontologies, e.g. we have at least a data complexity of coNP for a combination of normal logic programming rules with ontologies even if the data complexity of the Description Logics fragment is P. Indeed, although the approach of Hybrid MKNF knowledge bases is powerful, whenever we add rules with arbitrary nonmonotonic negation to an ontology, we loose in general tractability.

The reason for that lies in the way the reasoning algorithms work: models are in general guessed and then verified against several criteria yielding a lower bound on data complexity of at least NP. This is not surprising since [37] showed that there is a close correspondence between the logics of MKNF and Stable Model Semantics (SMS) [17] for logic programs whose reasoning algorithms are also based on guessing and checking models, so that any improvements based on SMS are rather unlikely. Instead, the other major semantics in Logic Programming (LP), the Well-Founded Semantics (WFS) [53], seems to offer a solution. WFS is a three-valued semantics allowing the truth values 'true', 'false' and 'undefined', while stable models only admit 'true' and 'false'. The value 'undefined' can be understood as delaying the decision whether some piece of information is true or false until clarifying information is available. This also permits to compute the unique well-founded model for a logic program in contrast to the possibly various stable models which have to be guessed. Though in general semantically weaker in terms of the derivable consequences, the lower complexity class (e.g. for normal programs the data complexity is P for the WFS instead of coNP for SMS [6]) makes WFS more promising for the intended application area, the World Wide Web, and applications like the one mentioned above ([44]) matching patient records for clinical trials with data of approximately 240,000 patients. In fact, the time required to answer example queries to such a large database in [44] varies from 26 minutes to more than 6 hours. According to [44] this is still within acceptable parameters but the addition of the desired non-monotonic feature as described above must avoid lifting the problem to a higher computational complexity, otherwise the solution is not feasable any longer. Additionally, SMS requires to obtain the entire model (just like [43] for combinations of rules and ontologies) for reasoning while the WFS permits top-down reasoning envolving only the part of the knowledge base relevant for the information of interest making WFS all the more suited for large scale applications. It would therefore be interesting to have a combination of rules and ontologies satisfying the 4 criteria mentioned above. Fortunately, there is a correspondence between Stable Model and Well-Founded semantics [52] relating the definition of the well-founded model to the operator used to verify stable model candidates and we want to explore this relation to adapt the approach presented in [43] to the WFS.

1.4. Contribution

In this paper, we thus define a new semantics, restricted to non-disjunctive rules, which soundly approximates the semantics of [43] but is in general in a strictly lower complexity class. In particular, when dealing with a tractable description logic our combined approach remains tractable w.r.t. data complexity. We extend the two-valued MKNF semantics from [43] to three truth values where each two-valued model from [41] corresponds to a total three-valued one of our approach (and vice versa) and where the least (w.r.t. derivable knowledge) three-valued MKNF model is the well-founded MKNF model. Our proposal satisfies straightforwardly the four criteria on a combination of rules and ontologies we presented above and the semantics guarantees the following properties:

- The well-founded MKNF model is faithful w.r.t. the two-valued models of [43], i.e. whatever is true, resp. false, in the well-founded MKNF model is also true, false resp., in each two-valued MKNF model as defined in [43].

- Our proposal yields the original DL-semantics when no rules are present, and the original WFS of logic programs if the DL component is empty.
- If the knowledge base is consistent then the approach is coherent in the sense of [45], i.e. whenever a formula is first-order false then it is enforced to be nonmonotonically false as well and not possibly undefined.
- If the knowledge base is inconsistent then our approach allows for detecting inconsistencies which might appear in tightly integrated ontologies and rules where information is spread out in both components of the knowledge base, e.g. if some formula is derivable from a rule to be true but also (first-order) false in the ontology. This detection is achieved without any substantial additional computational effort.
- The computational data complexity of our approach is dependent on the computational complexity of the applied DL but if the considered DL is of polynomial data complexity then the combination with rules remains polynomial.

1.5. Outline

The paper is structured as follows. We first recall preliminaries on Description Logics, the logics of Hybrid MKNF, and Hybrid MKNF knowledge bases in Section 2. Then we present the semantic framework in Section 3 which extends MKNF semantics to three truth values based on which the well-founded MKNF model is defined. In Section 4 we show how to obtain the well-founded MKNF model and also how the mentioned inconsistency detection works. It is also shown that the obtained result fits into the framework presented in Section 3. A comparison to related work follows in Section 5 before we conclude in Section 6 [7].

2. Preliminaries

In this section we recall preliminary notions for Description Logics, the logics of minimal knowledge and negation as failure, and Hybrid MKNF knowledge bases.

2.1. Description Logics

Our approach is basically independent of the underlying description logic. To make the exhibition self-contained, we recall the description logic $\mathcal{ALC}$, which is considered to be a foundational description logic for the research around OWL, and some standard extensions appearing for example in lightweight decription logics such as $\mathcal{EL}^{++}$ ([2]). This will suffice for the purpose of our paper, and

[7] A short version of this paper appeared in [33].

the reader familiar with description logics will have no difficulties adjusting our approach to more expressive description logics such as $\mathcal{SHOIN}$ or $\mathcal{SROIQ}$, which underlie OWL resp. OWL 2. For further background on description logics we refer to [3, 22].

The basic elements to represent knowledge in DLs are *individuals* that represent objects in a domain of discourse, *concepts* that group together individuals with common properties, and *roles* that put individuals in relation. The sets N_I, N_C and N_R of individual names, concept names and role names, respectively, form the basis to construct the syntactic elements of $\mathcal{ALC}$ according to the following grammar, in which $A \in N_C$ denotes an atomic concept, $C_{(i)}$ denote complex concepts, $r \in N_R$ denotes a role and $a_i \in N_I$ denote individuals.

$$C_{(i)} \quad \longrightarrow \quad \bot \mid \top \mid A \mid \neg C \mid C_1 \sqcap C_2 \mid C_1 \sqcup C_2 \mid \exists r.C \mid \forall r.C$$

The semantics of the syntactic elements of $\mathcal{ALC}$ is defined in terms of *interpretations* $I = (\Delta^I, \cdot^I)$ with a non-empty set Δ^I as the *domain* and an *interpretation function* $\cdot^I$ that maps each individual $a \in N_I$ to a distinct element $a^I \in \Delta^I$ and that interprets (possibly) complex concepts and roles as follows.

$$\begin{array}{rcl}
\top^I & = & \Delta^I \\
\bot^I & = & \emptyset \\
A^I & \subseteq & \Delta^I \\
r^I & \subseteq & \Delta^I \times \Delta^I \\
(C_1 \sqcap C_2)^I & = & C_1^I \cap C_2^I \\
(C_1 \sqcup C_2)^I & = & C_1^I \cup C_2^I \\
(\neg C)^I & = & \Delta^I \setminus C^I \\
(\forall r.C)^I & = & \{x \in \Delta^I \mid \forall y.(x,y) \in r^I \text{ implies } y \in C^I\} \\
(\exists r.C)^I & = & \{x \in \Delta^I \mid \exists y.(x,y) \in r^I \text{ and } y \in C^I\}
\end{array}$$

An $\mathcal{ALC}$ knowledge base $\mathcal{O}$ is a finite set of *axioms* formed by concepts, roles and individuals. A *concept assertion* is an axiom of the form $C(a)$ that assigns an individual a to a concept C. A *role assertion* is an axiom of the form $r(a_1, a_2)$ that relates two individuals a_1, a_2 by the role r. Concept and role assertions form the ABox. A *concept inclusion* is an axiom of the form $C_1 \sqsubseteq C_2$ that states the subsumption of the concept C_1 by the concept C_2, while a *concept equivalence* axiom $C_1 \equiv C_2$ is a shortcut for two inclusions $C_1 \sqsubseteq C_2$ and $C_2 \sqsubseteq C_1$. Concept inclusions and concept equivalences form the TBox. An interpretation I satisfies a concept assertion $C(a)$ if $a^I \in C^I$, a role assertion $r(a_1, a_2)$ if $(a_1^I, a_2^I) \in r^I$, a concept inclusion $C_1 \sqsubseteq C_2$ if $C_1^I \subseteq C_2^I$ and a concept equivalence $C_1 \equiv C_2$ if $C_1^I = C_2^I$. An interpretation that satisfies all axioms of a knowledge base $\mathcal{O}$ is called a *model* of $\mathcal{O}$. A concept C is called *satisfiable with respect to* $\mathcal{O}$ if $\mathcal{O}$ has a model in which $C^I \neq \emptyset$ holds.

$\mathcal{ALC}$ is a decidable logic; it is ExpTime-complete.

One common extension are role inclusion axioms appearing in the TBox. A *role inclusion* is an axiom of the form $r \sqsubseteq s$ that states the subsumption of two roles r and s. A *role composition* is an axiom of the form $r \circ s \sqsubseteq t$ that states the subsumption between a role composition $r \circ s$ and a role t. An interpretation I satisifies a role inclusion $r \sqsubseteq s$ if $r^I \sqsubseteq s^I$ and a role composition $r \circ s \sqsubseteq t$ if $\forall a_1, a_2, a_3 \in \Delta^I : (a_1, a_2) \in r^I \wedge (a_2, a_3) \in s^I \rightarrow (a_1, a_3) \in t^I$. We note that role compositions can be used to express transitivity of roles and left-and right-identity roles.

Example 1. We use the following example throughout the paper to illustrate the notions and definitions of our work. Note that we adopt the convention that names starting with a capital letter represent concepts/roles (termed DL-atoms below) and objects/individuals while names starting with a lower case letter represent variables and predicates not appearing in the ontology (non-DL atoms below).

$$\begin{aligned}
\mathrm{CD} &\sqsubseteq \exists \mathrm{HasPiece}.\mathrm{Piece} && (1)\\
\mathrm{Piece} &\sqsubseteq \exists \mathrm{HasArtist}.\mathrm{Artist} && (2)\\
\mathrm{HasPiece} \circ \mathrm{HasArtist} &\sqsubseteq \mathrm{HasArtist} && (3)\\
\mathrm{Topseller} \sqcup \mathrm{SpecialOffer} &\sqsubseteq \mathrm{Recommend} && (4)\\
\mathrm{HasPiece}(\mathrm{BNAW},\ \mathrm{BlueTrain}) & && (5)\\
\mathrm{HasArtist}(\mathrm{BlueTrain},\ \mathrm{JohnColtrane}) & && (6)
\end{aligned}$$

Consider an online store selling audio CDs. In order to attract more clients and raise sales, sophisticated tools for recommending CDs and searching for them shall be provided permitting the interaction of customized settings by the user with general guidelines by the company. For that purpose, an ontology is used for structuring and maintaining the database of CDs. Each CD is associated with a unique identifier, a publisher, a release date, and the pieces of music it contains. Each piece of music consists of at least one track but also permitting pieces with several tracks (as common for classical music). Additionally, each piece has its own unique identifier and can be associated with its artist, composer, genre, origin and so on. Axiom (1) states for example that each CD consists of at least one piece and axiom (2) expresses that each piece of music has an artist. We can also add role inclusions like (3) to formalize that if x is related to y by HasPiece and y related to z by HasArtist then x is related to z by HasArtist, i.e. HasArtist is a left-identity role. This enables us to derive from (1) - (3) that the artist of a piece on a certain CD is an artist of that CD. Note that this conclusion can be drawn without any present CDs, artists or pieces of music, as intended when reasoning with schema knowledge in an infinite domain. Of course, once specific information such as (5), i.e. the piece 'Blue Train' appears on the album 'The best Blue Note Album in the World... Ever' (BNAW), and (6), i.e. 'John Coltrane is the artist of Blue Train', is available, we are able to derive that John Coltrane appears as an artist on the album BNAW, and likewise for all the other artists appearing on this CD.

Moreover, we are able to express general guidelines for recommendations which the online store wants to apply. For example, CDs which are special offers or top sellers are automatically recommended to the customers (4).

2.2. Logics of Minimal Knowledge and Negation as Failure

The logic of minimal knowledge and negation as failure (MKNF) [37] extends first-order logic with two modal operators $\mathbf{K}$ and $\mathbf{not}$ which allow to inspect the knowledge base: intuitively, given a first-order formula φ, $\mathbf{K}\,\varphi$ asks whether φ is known while $\mathbf{not}\,\varphi$ is used to check whether φ is not known. The two modal operators thus allow local closed-word reasoning, in particular the operator $\mathbf{not}$ enables to draw conclusions from the absence of information, similar to default negation in Logic Programming. We present the syntax and the semantics of MKNF in the way introduced in [41, 43] and start with the syntax of MKNF formulas.

Let $\Sigma = (\Sigma_c, \Sigma_f, \Sigma_p)$ be a first-order signature, where Σ_c is a set of constants, Σ_f is a set of function symbols, and Σ_p is a set of predicates containing the binary equality predicate $\approx$. The syntax of MKNF formulas over Σ is defined as follows. A *first-order atom* $P(t_1, \ldots, t_n)$ is an MKNF formula where P is a predicate and the t_i are first-order terms. Sometimes, we also refer to such a predicate by P leaving the terms t_i implicit. If φ is an MKNF formula then $\neg\varphi$, $\exists x : \varphi$, $\mathbf{K}\,\varphi$ and $\mathbf{not}\,\varphi$ are MKNF formulas and likewise $\varphi_1 \wedge \varphi_2$ for MKNF formulas φ_1, φ_2. Moreover, $\varphi_1 \vee \varphi_2$, $\varphi_1 \supset \varphi_2$, $\varphi_1 \equiv \varphi_2$, $\forall x : \varphi$, $\mathbf{t}$, $\mathbf{f}$, $t_1 \approx t_2$ and $t_1 \not\approx t_2$ are abbreviations for $\neg(\neg\varphi_1 \wedge \neg\varphi_2)$, $\neg\varphi_1 \vee \varphi_2$, $(\varphi_1 \supset \varphi_2) \wedge (\varphi_2 \supset \varphi_1)$, $\neg(\exists x : \neg\varphi)$, $a \vee \neg a$, $a \wedge \neg a$, $\approx (t_1, t_2)$, and $\neg(t_1 \approx t_2)$. First-order atoms of the form $t_1 \approx t_2$ and $t_1 \not\approx t_2$ are called *equalities* and *inequalities*, respectively. Substituting the free variables x_i in φ, i.e. the variables which are not in the scope of any quantifier, by terms t_i is denoted $\varphi[t_1/x_1, \ldots, t_n/x_n]$. Given a (first-order) formula φ, $\mathbf{K}\,\varphi$ is called a *modal* $\mathbf{K}$*-atom* and $\mathbf{not}\,\varphi$ a *modal* $\mathbf{not}$*-atom*; modal $\mathbf{K}$-atoms and $\mathbf{not}$-atoms are *modal atoms*. If a modal atom does not occur in scope of a modal operator in an MKNF formula then it is *strict*. An MKNF formula φ without any free variables is *closed*, and it is *ground* if it does not contain variables at all. It is *modally closed* if all modal operators ($\mathbf{K}$ and $\mathbf{not}$) are applied in φ only to closed subformulas and *positive* if it does not contain the operator $\mathbf{not}$. An MKNF formula φ is *subjective* if all first-order atoms of φ occur within the scope of a modal operator and *flat* if it is subjective and all occurrences of modal atoms in φ are strict.

Let Σ be a signature and Δ a universe. Just like in first-order logic, a first-order interpretation I over Σ and Δ assigns an object $a^I \in \Delta$ to each constant $a \in \Sigma_c$, a function $f^I : \Delta^n \to \Delta$ to each n-ary function symbol $f \in \Sigma_f$, and a relation $P^I \subseteq \Delta^n$ to each n-ary predicate $P \in \Sigma_p$, and it interprets the predicate $\approx$ as equality - that is, for $\alpha, \beta \in \Delta$, we have $(\alpha, \beta) \in \approx^I$ iff $\alpha = \beta$. Unlike in standard first-order logic, for each element $\alpha \in \Sigma$, the signature Σ is required to contain a special constant n_α - called a *name* - such that $n_\alpha^I = \alpha$. The interpretation of a variable-free term $t = f(s_1, \ldots, s_n)$ is defined recursively as $t^I = f^I(s_1^I, \ldots, s_n^I)$.

The semantics of an MKNF formula over a signature Σ (henceforth considered implicit in all definitions) is defined as follows. An *MKNF structure*[8] is

[8]in [43] the term MKNF triple is used.

a triple (I, M, N) where I is a first-order interpretation over Δ and Σ, and M and N are nonempty sets of first-order interpretations over Δ and Σ. MKNF structures (I, M, N) define satisfiability of closed MKNF follows as follows:

$$\begin{array}{ll}
(I,M,N) \models P(t_1,\ldots,t_n) & \text{iff } (t_1^I,\ldots,t_n^I) \in P^I \\
(I,M,N) \models \neg\varphi & \text{iff } (I,M,N) \not\models \varphi \\
(I,M,N) \models \varphi_1 \wedge \varphi_2 & \text{iff } (I,M,N) \models \varphi_1 \text{ and } (I,M,N) \models \varphi_2 \\
(I,M,N) \models \exists x : \varphi & \text{iff } (I,M,N) \models \varphi[n_\alpha/x] \text{ for some } \alpha \in \Delta \\
(I,M,N) \models \mathbf{K}\,\varphi & \text{iff } (J,M,N) \models \varphi \text{ for all } J \in M \\
(I,M,N) \models \mathbf{not}\,\varphi & \text{iff } (J,M,N) \not\models \varphi \text{ for some } J \in N
\end{array}$$

Note that even though $\mathbf{K}$ and $\mathbf{not}$ are somehow related in the sense that $\mathbf{not}$ relates to $\neg\mathbf{K}$, in fact even identify if $M = N$, their evaluation is kept separate. The reason for that can be found in the nonmonotonic semantics presented in the following.

An *MKNF interpretation* M over a universe Δ is a nonempty set of first-order interpretations. For a closed MKNF formula φ, we say that M *satisfies* φ, i.e. $M \models \varphi$, if $(I, M, M) \models \varphi$ for each $I \in M$. This is still monotonic so we define a preference on those MKNF interpretations which satisfy the considered formula φ. An MKNF interpretation M over Δ is an *MKNF model* of a closed MKNF formula φ if (1) M satisfies φ and (2) for each MKNF interpretation M' such that $M' \supset M$ we have $(I', M', M) \not\models \varphi$ for some $I' \in M'$. An MKNF formula φ is *MKNF satisfiable* if an MKNF model exists; otherwise φ is *MKNF unsatisfiable*. Furthermore, φ *MKNF entails* ψ, written $\varphi \models_{MKNF} \psi$ if $M \models \psi$ for each MKNF model M of φ.

Example 2. Intuitively, an MKNF interpretation M is an MKNF model of an MKNF formula φ if M satisfies φ and if there is no superset M' of M which satisfies φ as well, using however M for the evaluation of modal $\mathbf{not}$-atoms. The definition is thus asymmetric w.r.t. $\mathbf{K}$ and $\mathbf{not}$: though $M = \{\{p\}\}$ satisfies both $\mathbf{K}\,p$ and $\neg\mathbf{not}\,p$, it is only an MKNF model for the first, not for the second since $(I', M', M) \models \neg\mathbf{not}\,p$ holds for any M' with $M' \supset M$.

The MKNF semantics as defined in [37] shows certain undesirable properties which is why [43] applies additionally the standard name assumption. We recall briefly the argument and the notion itself from [43].

One problem when using MKNF as in [37] for the integration of rules and ontologies is the usage of arbitrary universes. Let $\varphi = \varphi_1 \wedge \varphi_2$ where $\varphi_1 = \mathbf{K}\,A(A)$ and $\varphi_2 = \mathbf{not}\,A(B) \supset \mathbf{f}$. Intuitively, one would expect that φ is not satisfiable, however, if the universe contains only one element, then A and B are interpreted as the same object and φ is satisfied. We thus unintendedly derive that $\varphi \models A \approx B$, a conclusion we want to avoid in such cases. Another problem is caused by constants which are interpreted differently in different interpretations. Given $\varphi_1 = \mathbf{K}\,A(A)$ and $\varphi_2 = \exists x : \mathbf{K}\,A(x)$ we would expect that $\varphi_1 \models \varphi_2$. However, let M be an MKNF interpretation containing two elements I_1 and I_2 where each element maps A to a different name. Then, M

is an MKNF model of φ_1 but not for φ_2 since we need one domain element common to both interpretations.

Definition 1. (Standard Name Assumption [43]). A first-order interpretation I over a signature Σ employs the *standard name assumption* if

(1) the universe Δ of I contains all constants of Σ and a countably infinite number of additional constants called *parameters*,

(2) $t^I = t$ for each ground term t constructed using the function symbols from Σ and the constants from Δ, and

(3) the predicate $\approx$ is interpreted in I as a congruence relation - that is, it is reflexive, symmetric, transitive, and it allows the replacement of equals by equals [16].

Property (1) fixes the universe and property (2) makes constants rigid, i.e. assigns always the same domain elements to them, and together the two properties make each model equal to an Herbrand model with an infinite supply of constants. Property (3) itself treats $\approx$ not as true equality but as a congruence relation. The reason for that can be seen when considering $\varphi = \forall x : (x \approx A)$ and requiring (1) and (2) to hold in each interpretation. If $\approx$ is a true equality then (1) requires an infinite universe Δ but φ requires that Δ contains at most one element. Interpreting $\approx$ as a congruence relation enables a universe with infinitely many elements which are all congruent to each other.

It was shown, that consequences of first-order formulas under the standard first-order semantics and the standard name assumption cannot be distinguished. We thus use in the rest of the paper the standard name assumption for first-order inferences, but can consider them to be ordinary first-order differences, as we cannot tell the difference.

2.3. Hybrid MKNF Knowledge Bases

Hybrid MKNF knowledge bases as introduced in [41, 43][9] essentially are MKNF formulas restricted to a certain form. They consist of two components: a decidable description logic knowledge base translatable into first-order logic and a finite set of rules of modal atoms.

More precisely, the approach of hybrid MKNF knowledge bases is applicable to any first-order fragment $\mathcal{DL}$ satisfying the following conditions: (i) each knowledge base $\mathcal{O} \in \mathcal{DL}$ can be translated[10] into a formula $\pi(\mathcal{O})$ of function-free first-order logic with equality, (ii) it supports *A-Box*-assertions of the form $P(a_1, \ldots, a_n)$ for P a predicate and a_i constants of $\mathcal{DL}$ and (iii) satisfiability checking and instance checking (i.e. checking entailment of the form $\mathcal{O} \models P(a_1, \ldots, a_n)$) are decidable. In particular, description logics around OWL

[9] We will focus here on the presentation as in [41] and thus omit classical negation and arbitrary first-order formulas in rules in opposite to [43].

[10] See [3] for standard translations of Description Logic axioms.

satisfy these conditions. Note that we limit to function-free first order logic since otherwise decidability would not be possible anyway. Thus for the remainder of the paper, function symbols are not allowed in hybrid MKNF knowledge bases.

We recall MKNF rules and hybrid MKNF knowledge bases from [41].

Definition 2. Let $\mathcal{O}$ be a DL knowledge base. A function-free first-order atom $P(t_1, \ldots, t_n)$ over Σ such that P is $\approx$ or it occurs in $\mathcal{O}$ is called a *DL-atom*; all other atoms are called *non-DL-atoms*. An MKNF rule r has the following form where H_i, A_i, and B_i are function free first-order atoms:

$$\mathbf{K}\, H_1 \vee \ldots \vee \mathbf{K}\, H_l \leftarrow \mathbf{K}\, A_1, \ldots, \mathbf{K}\, A_n, \mathbf{not}\, B_1, \ldots, \mathbf{not}\, B_m \tag{7}$$

The sets $\{\mathbf{K}\, H_i\}$, $\{\mathbf{K}\, A_i\}$, and $\{\mathbf{not}\, B_i\}$ are called the *rule head*, the *positive body*, and the *negative body*, respectively. A rule r is *nondisjunctive* if $l = 1$; r is *positive* if $m = 0$; r is a *fact* if $n = m = 0$. A *program* $\mathcal{P}$ is a finite set of MKNF rules. A *hybrid MKNF knowledge base* $\mathcal{K}$ is a pair $(\mathcal{O}, \mathcal{P})$ and $\mathcal{K}$ is *nondisjunctive* if all rules in $\mathcal{P}$ are nondisjunctive.

We note that we will restrict our approach in the following sections to nondisjunctive rules while [41] includes disjunctions in the heads of rules.

MKNF rules are syntactically not precisely MKNF formulas so we have to transform them into MKNF formulas, respectively extend the transformation π which turns description logics into an MKNF formula. The straightforward extension is presented in the following.

Definition 3. Let $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ be a hybrid MKNF knowledge base. We extend π to rules r of the form (7), $\mathcal{P}$, and $\mathcal{K}$ as follows, where x is the vector of the free variables of r.

$$\pi(r) = \forall x : (\mathbf{K}\, H_1 \vee \ldots \vee \mathbf{K}\, H_l \subset \mathbf{K}\, A_1 \wedge \ldots \wedge K A_n \wedge \mathbf{not}\, B_1 \wedge \ldots \wedge \mathbf{not}\, B_m)$$

$$\pi(\mathcal{P}) = \bigwedge_{r \in \mathcal{P}} \pi(r) \qquad \pi(\mathcal{K}) = \mathbf{K}\, \pi(\mathcal{O}) \wedge \pi(\mathcal{P})$$

Even hybrid MKNF knowledge bases without function symbols are in general undecidable unless restricted in some way because rules can be applied to all the objects in the infinite domain. The basic idea to make reasoning with hybrid MKNF knowledge bases decidable is to apply rules only to individuals which appear in the knowledge base. This restriction is achieved by DL-safety.

Definition 4. An MKNF rule r is *DL-safe* if every variable in r occurs in at least one non-DL-atom $\mathbf{K}\, B$ occurring in the body of r. A hybrid MKNF knowledge base $\mathcal{K}$ is *DL-safe* if all its rules are DL-safe.

In the following we suppose that all MKNF knowledge bases are DL-safe.

Then, grounding the knowledge base ensures that all the individuals appearing in it are applicable to rules and DL-safety guarantees that no other individual can be used.

Definition 5. Given a hybrid MKNF knowledge base $\mathcal{K} = (\mathcal{O}, \mathcal{P})$, the *ground instantiation of* $\mathcal{K}$ is the KB $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ where $\mathcal{P}_G$ is obtained from $\mathcal{P}$ by replacing each rule r of $\mathcal{P}$ with a set of rules substituting each variable in r with constants from $\mathcal{K}$ in all possible ways.

It was shown in [43], for a DL-safe hybrid KB $\mathcal{K}$ that the MKNF models of $\mathbf{K}$ and $\mathcal{K}_G$ coincide.

Several reasoning algorithms were provided for combinations of arbitrary description logic fragments and rules of differing expressivity in [43] and their corresponding data complexities, and some of those we recall in the following table. The table mentions the results for combinations with nondisjunctive rules (with or without **not** in the rules) and description logics fragments of differing computational complexity for instance checking.

not	$\mathcal{DL} = \emptyset$	$\mathcal{DL} \in$ P	$\mathcal{DL} \in$ coNP
no	P	P	coNP
yes	coNP	coNP	Π_2^p

We point out that allowing **not**, i.e. nonmonotonic negation, increases data complexity drastically and in particular beyond tractability. In the following sections we will provide a semantics and its computation which improves this issue substantially, but before coming to that, we finish this section with the continuation of the running example related to recommending CDs.

Example 3. Consider Example 1 where we show some ontology axioms to represent the schema knowledge for audio CDs and a guideline for recommending CDs from the store. Now we want to present a hybrid MKNF knowledge base $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ which enables recommending CDs based on the guidelines from the company and the customer-specific settings. Therefore, we suppose that the ontology contains the statements shown in Example 1 and similar statements such as 'every piece has a composer' which are only mentioned in the example but not explicitly listed. Now, let us add some rules ro realize personalized recommendations.

$$\mathbf{K}\,\mathrm{Recommend(x)} \leftarrow \mathbf{K}\,\mathrm{CD(x)}, \mathbf{not}\,\mathrm{owns}(x), \mathbf{not}\,\mathrm{LowEval}(x), \mathbf{K}\,\mathrm{interesting}(x). \quad (8)$$

$$\mathbf{K}\,\mathrm{interesting}(x) \leftarrow \mathbf{K}\,\mathrm{CD}(x), \mathbf{K}\,\mathrm{CD}(y), \mathbf{K}\,\mathrm{owns(y)}, \mathbf{not}\,\mathrm{owns}(x), \mathbf{K}\,\mathrm{similar}(x,y). \quad (9)$$

$$\mathbf{K}\,\mathrm{similar}(x,y) \leftarrow \mathbf{K}\,\mathrm{CD(x)}, \mathbf{K}\,\mathrm{CD(y)}, \mathbf{K}\,\mathrm{Artist(z)}, \mathbf{K}\,\mathrm{HasArtist}(x,z), \mathbf{K}\,\mathrm{HasArtist}(y,z). \quad (10)$$

$$\mathbf{K}\,\mathrm{owns(EnConcert)} \leftarrow \quad (11)$$

$$\mathbf{K}\,\mathrm{HasArtist(EnConcert,JackJohnson)} \leftarrow \quad (12)$$

$$\mathbf{K}\,\mathrm{HasArtist(ByTheSea,JackJohnson)} \leftarrow \quad (13)$$

$$\mathbf{K}\,\mathrm{SpecialOffer(BNAW)} \leftarrow \quad (14)$$

Imagine that we want to get recommendations for interesting CDs which we do not own and which do not get a low evaluation (8). Note that we use closed-world reasoning for owns and lowEval. In case of owns it is reasonable to assume that the knowledge about owned CDs is fully available. In case of lowEval it might happen that there is no evaluation yet available and we want the recommendation anyway. A CD shall only not be considered for recommendation if there actually is a known low evaluation for it. This evaluation could be taken from other customers of the store or from a web page of professional reviews, here, for simplicity, we keep this part of the reasoning process implicit. A CD could be interesting if there is a another similar CD wich is already owned (9) and two CDs are similar if e.g. their artists correspond (10). Note that e.g. the predicate CD is used to ensure DL-safety, and we assume that the instances of that predicate relevant for any drawn conclusion are always appropriately defined. If we now add the facts (11) - (14) then we can derive Recommend(ByTheSea), since no low evaluation is known for that CD, and Recommend(BNAW) as it is a special offer. Hybrid MKNF knowledge bases thus allow us to obtain consequences for predicates which are 'defined' both in the ontology and in the rules. The result might then be further applied to derive subsequent consequences either in rules or in the ontology without any problems. Note that the facts (12) and (13) are here explicitly added, representing the implicit consequences derivable from the appropriate ontology alone, similar to HasArtist(BNAW,JohnColtrane) in Example 1.

3. Three-valued MKNF Semantics

In this section, we introduce a three-valued semantics for hybrid MKNF knowledge bases. The rationale behind it, and main motivation, is to obtain a semantics which is related to the well-founded semantics in order to take advantage of its (data) complexity which is lower than the (data) complexity of the corresponding two-valued semantics which is based on stable models. Nevertheless, the DL-part of a hybrid MKNF knowledge base is desirably still interpreted under two-valued semantics, simply because this is sufficient to obtain the desired complexity results, and obviously it is advantageous to change as little as possible.

3.1. Evaluation in MKNF Structures

The two-valued hybrid MKNF semantics [43] is closely related [37] to stable models [17] and both semantics permit in general for several models. In fact, MKNF formulas such as $\varphi = ((\mathbf{not}\, p \supset \mathbf{K}\, q \wedge (\mathbf{not}\, q \supset \mathbf{K}\, p))$ (and its corresponding set of rules) have two models - one in which p is true and q is false, and one in which these truth values are inverse. Choices like this one then enforce to guess models resulting in the higher computational complexity. One way of generalizing this semantic framework is known from Logic Programming with the three-valued semantics, such as the well-founded semantics [53], which relies on three truth values. Intuitively, a third truth value $\mathbf{u}$, denoting undefined, is introduced as an alternative to the values **t** and **f**, permitting to delay the evaluation to any of the two latter values until further information is available. While the well-founded semantics avoids as many choices as possible, a further generalization, partial stable models [46], allows three-valued models with varying undefinedness, thus generalizing stable models and well-founded semantics. When defining a three-valued MKNF semantics we want to follow this idea. There is however one more problem to be taken into account: since we are interested in applying the semantics also to hybrid MKNF knowledge bases, which contain two-valued ontologies, we want to define the semantics in such a way that an MKNF formula corresponding to a description logics fragment is ensured to be two-valued. In particular, we want the semantics to coincide with two-valued (first-order) semantics for a formula without modal operators, such as a description logics knowledge base translatable into an MKNF formula.

We therefore define in the following a three-valued MKNF semantics which extends the two-valued one but remains two-valued in case of MKNF formulas without modal operators. We start by defining MKNF structures.

Definition 6. A *three-valued (partial) MKNF structure* $(I, \mathcal{M}, \mathcal{N})$ consists of a first-order interpretation I and two pairs $\mathcal{M} = \langle M, M_1 \rangle$ and $\mathcal{N} = \langle N, N_1 \rangle$ of sets of first-order interpretations where $M_1 \subseteq M$ and $N_1 \subseteq N$. It is called *total* if $\mathcal{M} = \langle M, M \rangle$ and $\mathcal{N} = \langle N, N \rangle$.

In the two-valued semantics, an MKNF structure (I, M, N) contains sets of interpretations M and N for evaluating a modal atom $\mathbf{K}\,\varphi$, respectively $\mathbf{not}\,\varphi$, to **t** or **f**, depending on whether φ is contained in all elements of M, respectively N, or not. This however leaves no space for an extension to a third truth value **u**. So, we turn sets of interpretations into pairs of sets of interpretations. Then, as we will see below, e.g. a modal atom $\mathbf{K}\,\varphi$ is true w.r.t. $\langle M, M_1 \rangle$ if φ is true in all elements of M; if φ is not true in all elements of M then it is undefined in case φ is true in all elements of M_1 or false otherwise. The additional restrictions, saying that $M_1 \subseteq M$ and $N_1 \subseteq N$, in fact ensure that no modal atom can be both true and false at the same time, and it can easily be shown via induction that the same holds for any MKNF formula φ. We guarantee this way that no fourth truth value 'both' is needed. Nevertheless, given an MKNF formula φ, partial MKNF structures may evaluate $\mathbf{K}\,\varphi$ and $\mathbf{not}\,\varphi$ to true at the same time, just like in the two-valued case and we will show below how to prevent this from happening when defining interpretation pairs.

We now present the evaluation of closed MKNF formulas in such three-valued MKNF structures.

Definition 7. Let $(I, \mathcal{M}, \mathcal{N})$ be a three-valued MKNF structure and $\{\mathbf{t}, \mathbf{u}, \mathbf{f}\}$ the set of truth values with the order $\mathbf{f} < \mathbf{u} < \mathbf{t}$, where the operator max (resp. min) chooses the greatest (resp. least) element with respect to this ordering. We define:

- $(I, \mathcal{M}, \mathcal{N})(P(t_1, \ldots, t_n)) = \begin{cases} \mathbf{t} & \text{iff } P(t_1, \ldots, t_n) \in I \\ \mathbf{f} & \text{iff } P(t_1, \ldots, t_n) \notin I \end{cases}$

- $(I, \mathcal{M}, \mathcal{N})(\neg\varphi) = \begin{cases} \mathbf{t} & \text{iff } (I, \mathcal{M}, \mathcal{N})(\varphi) = \mathbf{f} \\ \mathbf{u} & \text{iff } (I, \mathcal{M}, \mathcal{N})(\varphi) = \mathbf{u} \\ \mathbf{f} & \text{iff } (I, \mathcal{M}, \mathcal{N})(\varphi) = \mathbf{t} \end{cases}$

- $(I, \mathcal{M}, \mathcal{N})(\varphi_1 \wedge \varphi_2) = \min\{(I, \mathcal{M}, \mathcal{N})(\varphi_1), (I, \mathcal{M}, \mathcal{N})(\varphi_2)\}$

- $(I, \mathcal{M}, \mathcal{N})(\varphi_1 \supset \varphi_2) = \mathbf{t}$ iff $(I, \mathcal{M}, \mathcal{N})(\varphi_2) \geq (I, \mathcal{M}, \mathcal{N})(\varphi_1)$ and $\mathbf{f}$ otherwise

- $(I, \mathcal{M}, \mathcal{N})(\exists x : \varphi) = \max\{(I, \mathcal{M}, \mathcal{N})(\varphi[\alpha/x]) \mid \alpha \in \Delta\}$

- $(I, \mathcal{M}, \mathcal{N})(\mathbf{K}\,\varphi) = \begin{cases} \mathbf{t} & \text{iff } (J, \langle M, M_1 \rangle, \mathcal{N})(\varphi) = \mathbf{t} \text{ for all } J \in M \\ \mathbf{f} & \text{iff } (J, \langle M, M_1 \rangle, \mathcal{N})(\varphi) = \mathbf{f} \text{ for some } J \in M_1 \\ \mathbf{u} & \text{otherwise} \end{cases}$

- $(I, \mathcal{M}, \mathcal{N})(\mathbf{not}\,\varphi) = \begin{cases} \mathbf{t} & \text{iff } (J, \mathcal{M}, \langle N, N_1 \rangle)(\varphi) = \mathbf{f} \text{ for some } J \in N_1 \\ \mathbf{f} & \text{iff } (J, \mathcal{M}, \langle N, N_1 \rangle)(\varphi) = \mathbf{t} \text{ for all } J \in N \\ \mathbf{u} & \text{otherwise} \end{cases}$

As intended, this evaluation is not a purely three-valued one, since first-order atoms are evaluated like in the two-valued case. In fact, an MKNF formula without modal operators, and thus also a pure description logics knowledge base, is only two-valued and it can easily be seen that it is evaluated in exactly the same way as in the scheme presented in Section 2. This is desired in particular when the knowledge base consists just of the DL part. So, the third truth value only affects MKNF formulas containing modal atoms, which in case of hybrid MKNF knowledge bases limits the occurrence to rules. These rules, corresponding to implications, are, however, no longer interpreted classically, in a first-order sense: $\mathbf{u} \leftarrow \mathbf{u}$ is true in the evaluation presented above while the classical boolean correspondence $\mathbf{u} \vee \neg\mathbf{u}$, respectively $\neg(\neg\mathbf{u} \wedge \mathbf{u})$, is undefined. The reason for this change is that, this way, rules can only be true or false, similarly to what happens in Logic Programming, even though they might contain undefined modal atoms. Intuitively, the advantage for hybrid MKNF knowledge bases is that we can leave single modal atoms undefined, thus not necessarily having to create several models, while the entire knowledge base is only true or false.

We point out that the evaluation of **not** w.r.t. $\langle N, N_1 \rangle$ is symmetrical to the evaluation of $\mathbf{K}$ w.r.t. $\langle M, M_1 \rangle$, only that, just like the two-valued evaluation in Section 2, the conditions are switched. E.g. the condition for true modal $\mathbf{K}$-atoms yields false modal **not**-atoms w.r.t. N. This is identical to the two-valued (monotonic) evaluation and is justified by the intended correspondence between **not** and $\neg\mathbf{K}$ mentioned in Section 2.

3.2. Partial MKNF Models

In the following we will extend the two-valued MKNF interpretations, since one set of first-order interpretations is not sufficient to represent truth, falsity, and undefinedness of modal atoms the way we introduced evaluation beforehand.

Definition 8. An *interpretation pair* (M, N) consists of two MKNF interpretations M, N with $\emptyset \subset N \subseteq M$. An interpretation pair *satisfies* a closed MKNF formula φ, written $(M, N) \models \varphi$, if and only if $(I, \langle M, N\rangle, \langle M, N\rangle)(\varphi) = \mathbf{t}$ for each $I \in M$. If $M = N$ then the interpretation pair (M, N) is called *total*. If there exists an interpretation pair satisfying φ then φ is *consistent*.

The set M contains all interpretations which model only truth while N models everything which is true or undefined. Evidently, just as in the two-valued case, anything not being modeled in N is false. The included subset relation between M and N ensures that interpretation pairs are defined in accordance with three-valued MKNF structures so that each formula is evaluated to exactly one truth value. Note the striking similarity compared to MKNF interpretations in the two-valued case by using the interpretation pair (M, N) to evaluate both $\mathbf{K}$ and $\mathbf{not}$ simultaneously.

Now we define the preference relation, i.e. the model notion, on interpretation pairs similar in spirit to the two-valued case, i.e. minimizing truth and undefinedness of formulas w.r.t. $\mathbf{K}$.

Definition 9. Any interpretation pair (M, N) is a *partial (three-valued) MKNF model* for a given closed MKNF formula φ if

(1) (M, N) satisfies φ and

(2) for each interpretation pair (M', N') with $M \subseteq M'$ and $N \subseteq N'$, where at least one of the inclusions is proper and $M' = N'$ if $M = N$, there is $I' \in M'$ such that $(I', \langle M', N'\rangle, \langle M, N\rangle)(\varphi) \neq \mathbf{t}$.

Condition (1) checks whether (M, N) evaluates φ to $\mathbf{t}$ while the second condition verifies that (M, N) contains only knowledge necessary to obtain this evaluation to $\mathbf{t}$. This is achieved in a way similar to the two-valued MKNF semantics: it is checked for each interpretation pair (M', N') which intuitively properly subsumes (M, N) that φ does not evaluate to $\mathbf{t}$ for all $I' \in M'$, where (M', N') is used to evaluate $\mathbf{K}$ while (M, N) evaluates $\mathbf{not}$. Intuitively, one may consider an interpretation pair as a guess for the true evaluation of the considered formula and condition (2) checks, having fixed the evaluation of modal $\mathbf{not}$-atoms, whether the evaluation of modal $\mathbf{K}$-atoms is actually minimal w.r.t. to the order $\mathbf{f} < \mathbf{u} < \mathbf{t}$ of truth values. We will use an example to demonstrate how this minimization of derived knowledge is achieved.

Example 4. Consider the following MKNF formula φ corresponding to two rules.

$$(\mathbf{not}\, p \supset \mathbf{K}\, q) \wedge (\mathbf{not}\, q \supset \mathbf{K}\, p)$$

A interpretation pair (M, N) which satisfies condition (1) of Definition 9 has to evaluate both conjuncts to true. The interpretation pair $(\{\{p\}, \{p, q\}\}, \{\{p, q\}\})$ which evaluates $\mathbf{K}\, p$ to $\mathbf{t}$ and $\mathbf{K}\, q$ to $\mathbf{u}$ satisfies the first condition but is not an MKNF model since e.g. $(M', N') = (\{\{p\}, \{p, q\}\}, \{\{p, q\}\})$ violates condition

(2). In fact, since **not** is always evaluated with respect to (M, N), the two rules are true anyway and, for $N = \{\{p, q\}\}$, the only MKNF model is the one with $M = \{\emptyset, \{p\}, \{q\}, \{p, q\}\}$, i.e. the interpretation pair which evaluates $\mathbf{K}\, p$ and $\mathbf{K}\, q$ to $\mathbf{u}$. In other words, the initial interpretation pair was not minimal w.r.t. the evaluation of modal $\mathbf{K}$-atoms. Similar to the minimization of the evaluation of $\mathbf{K}\, p$ from $\mathbf{t}$ to $\mathbf{u}$, alterations from $\mathbf{u}$ to $\mathbf{f}$ are possible: maintain the original $M = \{\{p\}, \{p, q\}\}$ and set $N = M$. Now the evaluation of $\mathbf{K}\, q$ is minimized from $\mathbf{u}$ to $\mathbf{f}$ and it is easy to verify that the resulting interpretation pair is in fact an MKNF model.

It should be pointed out that a larger set M or N means less true or undefined knowledge, i.e. minimization is achieved by increasing the sets in consideration. Note that $N' \subseteq M'$ for interpretation pairs (M', N') ensures that we only check reasonable candidates for augmenting (M, N)[11].

We now adapt some notions needed in the remainder, such as consistency of MKNF formulas, from the two-valued MKNF semantics to the three-valued setting.

Definition 10. If there is a partial MKNF model for a given closed MKNF formula φ then φ is called *MKNF-consistent*, otherwise it is called *MKNF-inconsistent*. If $(I, \langle M, N\rangle, \langle M, N\rangle)(\psi) = \mathbf{t}$ for all partial MKNF models (M, N) of φ then φ *entails* ψ, written $\varphi \models^3_{MKNF} \psi$.

Though the notions of consistency and satisfiability are usually applied in the same technical sense, we want to distinguish between MKNF-satisfiability in the two-valued case and MKNF-consistency for three-valued models. Likewise, we distinguish between the two-valued notion 'MKNF entails' and the three-valued one 'entails'.

In spite of keeping notions separate, two- and three-valued MKNF models are closely related: we will now show that any (two-valued) MKNF model M corresponds exactly to a (total) three-valued one and vice-versa. For that purpose, we first prove that evaluation in MKNF structures (I, M, N) and total three-valued structures $(I, \langle M, M\rangle, \langle N, N\rangle)$ is identical due to the fact that nothing can be undefined in such a three-valued structure.

Lemma 1. *Given a closed MKNF formula φ, $(I, M, N) \models \varphi$ if and only if $(I, \langle M, M\rangle, \langle N, N\rangle)(\varphi) = \mathbf{t}$.*

Proof. The proof is done by induction on the formula φ.

Let φ be $P(t_1, \ldots, t_n)$. We have $(I, M, N) \models P(t_1, \ldots, t_n)$ iff $P(t_1, \ldots, t_n) \in I$ iff $(I, \langle M, M\rangle, \langle N, N\rangle)(P(t_1, \ldots, t_n)) = \mathbf{t}$.

[11] In comparison to [33] the definition has been slightly altered to simplify proofs and computation: in case of a total interpretation pair (M, M) it is sufficient to check that no other total interpretation pair (M', M') actually yields a true evaluation for all $I' \in M'$. This simplification is also justified by the intuition of enlarging N' separately: there is no undefinedness in a total interpretation pair, and minimization of undefinedness is thus not necessary.

Assume that the lemma holds for φ_1. We show the induction steps for $\neg$ and $\mathbf{K}$, all the other cases follow analogously.

Let φ be $\neg\varphi_1$. We have that $(I, M, N) \models \neg\varphi_1$ iff $(I, M, N) \not\models \varphi_1$ iff, by induction hypothesis, $(I, \langle M, M\rangle, \langle N, N\rangle)(\varphi_1) = \mathbf{f}$ iff by definition of evaluation in partial structures $(I, \langle M, M\rangle, \langle N, N\rangle)(\neg\varphi_1) = \mathbf{t}$.

Let φ be $\mathbf{K}\,\varphi_1$. We have $(I, M, N) \models \mathbf{K}\,\varphi_1$ iff $(I, M, N) \models \varphi_1$ holds for each $I \in M$ iff $(I, \langle M, M\rangle, \langle N, N\rangle)(\varphi_1) = \mathbf{t}$ for all $I \in M$ by induction hypothesis iff $(I, \langle M, M\rangle, \langle N, N\rangle)(\mathbf{K}\,\varphi_1) = \mathbf{t}$. $\square$

This lemma can be used to show the following proposition which not only states that every two-valued model M corresponds to a three-valued one (M, M) like in [32] but also the other way around, i.e. every total model (M, M) corresponds to a two-valued one in the sense of [43].

Proposition 1. *Given a closed MKNF formula φ, M is an MKNF model of φ if and only if (M, M) is a three-valued MKNF model of φ.*

Proof. Let (M, M) be a three-valued MKNF model of φ, i.e. (M, M) satisfies the two conditions of Definition 9. We show that M is a (two-valued) MKNF model. It follows from the first of the two conditions of Definition 9 that $(I, \langle M, M\rangle, \langle M, M\rangle)(\varphi) = \mathbf{t}$ for all $I \in M$ and therefore, by Lemma 1, that $(I, M, M) \models \varphi$ for each $I \in M$. The second condition states for each interpretation pair (M', M'), with $M \subset M'$ that we have $(I', \langle M', M'\rangle, \langle M, M\rangle)(\varphi) \neq \mathbf{t}$ for some $I' \in M'$. We conclude from Lemma 1 that for any M' with $M' \supset M$ there is an $I' \in M'$ such that $(I', M', M) \not\models \varphi$.

Now, let M be a two-valued MKNF model of φ, we show that (M, M) is a three-valued MKNF model. We know that $(I, M, M) \models \varphi$ for each $I \in M$ since M is a two-valued MKNF model. Then $(I, \langle M, M\rangle, \langle M, M\rangle)(\varphi) = \mathbf{t}$ holds for all $I \in M$ by Lemma 1, so the first of the two conditions of Definition 9 is satisfied. Furthermore, since M is an MKNF model of φ, we know that for all M' with $M' \supset M$ we have $(I', M', M) \not\models \varphi$ for some $I' \in M'$. Again, from Lemma 1, we know that for any interpretation pair (M', M') with $M' \supset M$ we have $(I', \langle M', M'\rangle, \langle M, M\rangle)(\varphi) \neq \mathbf{t}$ for some $I' \in M'$. This is sufficient since, according to Definition 9, for (M, M) we only consider need to consider total interpretation pairs (M', M'). $\square$

We are able to compare interpretation pairs based on an order that resembles the knowledge order from Logic Programming. Intuitively, given such an order and two interpretation pairs (M_1, N_1) and (M_2, N_2), we have that (M_1, N_1) is greater than (M_2, N_2) w.r.t. such an order if (M_1, N_1) allows to derive more true and false knowledge than (M_2, N_2). Taking into account that a larger set of interpretations permits the derivation of less true and more false knowledge, we define:

Definition 11. Let (M_1, N_1) and (M_2, N_2) be interpretation pairs. We have that $(M_1, N_1) \succeq_k (M_2, N_2)$ iff $M_1 \subseteq M_2$ and $N_1 \supseteq N_2$.

Such an order is of particular interest for comparing models. In fact, in Logic Programming the least model among all partial models for a given program is the well-founded model. Similarly, here we want to introduce a notion referring to the least partial MKNF model, i.e. the one among all partial models which leaves as much as possible undefined.

Definition 12. Let φ be a closed MKNF formula and (M, N) be a partial MKNF model of φ such that $(M_1, N_1) \succeq_k (M, N)$ for all partial MKNF models (M_1, N_1) of φ. Then (M, N) is a *well-founded MKNF model*.

Of course, if φ is inconsistent then there are no partial MKNF models and thus no well-founded MKNF model. However, in case of consistency, we will obtain that, if φ is a hybrid MKNF knowledge base, the unique well-founded MKNF model exists. This model is especially important, in that, as we shall see, a modal atom is true in it iff it is true in all partial MKNF models. This way, performing skeptical reasoning in partial MKNF models amounts to determining the well-founded MKNF model.

Theorem 1. *Let $\mathcal{K}$ be a consistent nondisjunctive DL-safe hybrid MKNF KB. Then the well-founded MKNF model exists.*

The respective proofs for the uniqueness/existence of the well-founded MKNF model and how to calculate it follow in section 4 (as a direct consequence of Theorem 5). The following example gives at least an intuitive insight to the correspondence of two-valued and three-valued MKNF models and the well-founded MKNF model.

Example 5. Consider the knowledge base $\mathcal{K}$ corresponding to the MKNF formula φ from Example 4.

$$\begin{aligned} \mathbf{K}\, q &\leftarrow \mathbf{not}\, p \\ \mathbf{K}\, p &\leftarrow \mathbf{not}\, q \end{aligned}$$

The (two-valued) MKNF models are $\{\{p\}, \{p, q\}\}$ and $\{\{q\}, \{p, q\}\}$, i.e. $\mathbf{K} p$ and $\mathbf{not} q$ are true in the first model, and $\mathbf{K} q$ and $\mathbf{not} p$ are true in the second one. We thus obtain two total three-valued MKNF models: $(\{\{p\}, \{p, q\}\}, \{\{p\}, \{p, q\}\})$ and $(\{\{q\}, \{p, q\}\}, \{\{q\}, \{p, q\}\})$. In addition, $(\{\emptyset, \{p\}, \{q\}, \{p, q\}\}, \{\{p, q\}\})$ is the only other MKNF model as we have already seen in Example 4. This MKNF model satisfies the condition given in Definition 12 and is thus the well-founded MKNF model of $\mathcal{K}$ as we will see in the following section.

In the remainder of this section we show two further properties which were already proven to hold for the two-valued case in [40] but since our semantics differs, we lift these important statements to the three-valued semantics.

The first property states that $\mathbf{K}$ can be introduced in front of an arbitrary closed MKNF formula without changing the models satisfying that formula.

Proposition 2. *Let σ be a closed MKNF formula and let (M, N) be an interpretation pair. Then, (M, N) is an MKNF model of σ if and only if (M, N) is an MKNF model of $\mathbf{K}\,\sigma$.*

Proof. Suppose that (M, N) is an MKNF model of σ. We know for all $I \in M$ that $(I, \langle M, N\rangle, \langle M, N\rangle)(\sigma) = \mathbf{t}$, so $(I, \langle M, N\rangle, \langle M, N\rangle)(\mathbf{K}\ \sigma) = \mathbf{t}$ holds for all $I \in M$ as well. Since for each (M', N') there is $I' \in M'$ such that $(I', \langle M', N'\rangle, \langle M, N\rangle)(\sigma) \neq \mathbf{t}$ we also obtain the same for $\mathbf{K}\,\sigma$ and (M, N) is an MKNF model of $\mathbf{K}\,\sigma$. The converse direction follows analogously. □

The second property we lift says that grounding a hybrid MKNF knowledge base does not affect the partial MKNF models. This shows that $\mathcal{K}$ and its grounded version derive the same consequences.

Lemma 2. *Let $\mathcal{K}$ be a DL-safe hybrid MKNF knowledge base and ψ a ground MKNF formula. Then $\pi(\mathcal{K}) \models^3_{MKNF} \psi$ if and only if $\pi(\mathcal{K}_G) \models^3_{MKNF} \psi$.*

Proof. The argument showing the contrapositive statement $\pi(\mathcal{K}) \not\models^3_{MKNF} \psi$ if and only if $\pi(\mathcal{K}_G) \not\models^3_{MKNF} \psi$ is absolutely identical to the one in [40]. So we simply refer to the proof given there. □

4. The Well-founded MKNF Model

In this section we are going to define the procedure for computing the unique model for hybrid MKNF knowledge bases, i.e. the well-founded MKNF model, outlined in the end of the previous section. This procedure is obtained by adapting the alternating fixpoint construction for the Well-Founded Semantics [52] from Logic Programming to hybrid MKNF knowledge bases taking into account possible conflicts resulting from the combination of classical negation in ontologies and nonmonotonic negation in rules.

We start by adapting from [43] a means for representing interpretation pairs in a more simple way. Then, based on that means, we define operators which allow us to compute a unique model for hybrid MKNF knowledge bases. We show that this model is indeed the well-founded MKNF model, and we present several important properties including its computational complexity, faithfulness w.r.t. to the Well-Founded Semantics for Logic Programming, and discovery of inconsistencies.

4.1. Partitions of Modal Atoms

As argued in [43], since the MKNF models of arbitrary hybrid MKNF knowledge bases with a countably infinite domain are infinite, working with MKNF models is cumbersome. The same holds for interpretation pairs in three-valued semantics presented in Section 3, so some finite representation is required. The solution, applied in [43] and originally from [10], is to represent MKNF models by a finite first-order formula whose set of (first-order) models corresponds to the (two-valued) MKNF model. Intuitively, such a first-order formula is obtained in [43] by first dividing the modal atoms occurring in the ground hybrid

MKNF knowledge base into true and false modal atoms, and then constructing the first-order formula from the true modal atoms and the ontology. We extend this construction and its related notions from [43] to three truth values, i.e. a partition into three disjoint sets.

Definition 13. Let $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ be a ground hybrid MKNF knowledge base. The *set of* $\mathbf{K}$*-atoms of* $\mathcal{K}_G$, written $\mathsf{KA}(\mathcal{K}_\mathsf{G})$, is the smallest set that contains (i) all ground $\mathbf{K}$-atoms occurring in $\mathcal{P}_G$, and (ii) a modal atom $\mathbf{K}\,\xi$ for each ground modal atom $\mathbf{not}\,\xi$ occurring in $\mathcal{P}_G$. A *partial partition* (T, F) of $\mathsf{KA}(\mathcal{K}_G)$ consists of three sets, namely T, F, and $U = \mathsf{KA}(\mathcal{K}_G) \setminus (T \cup F)$ where $T, F \subseteq \mathsf{KA}(\mathcal{K}_G)$ and $T \cap F = \emptyset$.

The set $\mathsf{KA}(\mathcal{K}_G)$ contains all modal atoms occurring in $\mathcal{K}_G$ only that modal **not**-atoms are substituted by corresponding modal $\mathbf{K}$-atoms. That set can be partitioned into two sets T and F which are disjoint, but their union does not neccessarily equal to $\mathsf{KA}(\mathcal{K}_G)$ permitting a third set U. Intuitively, T contains true modal atoms, F the false modal atoms, and U all the remaining which are considered to be undefined.

In [43], a set of first-order formulas is defined from such a set of true modal atoms and the ontology with the aim of using this set of formulas to represent the knowledge base $\mathcal{K}_G$. Then, the set of first-order interpretations satifying that set of formulas corresponds to one MKNF model of $\mathcal{K}_G$.

Here, this construction will not suffice and we show below how we adapt this idea to a three-valued setting. The definition of such a set of first-order formulas can be recalled from [43].

Definition 14. Let $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ be a ground hybrid MKNF knowledge base. For a subset S of $\mathsf{KA}(\mathcal{K}_\mathsf{G})$, the *objective knowledge* of S w.r.t. $\mathcal{K}_G$ is the set of first-order formulas $\mathsf{OB}_{\mathcal{O},S} = \{\pi(\mathcal{O})\} \cup \{\xi \mid \mathbf{K}\,\xi \in S\}$.

This notion will be used to establish a link between partial MKNF models and partial partitions. For that purpose we need to adopt one more notion from [43].

Definition 15. Let S be a set of ground modal $\mathcal{K}$-atoms. A partial partition (T, F) of S is *induced by* an interpretation pair (M, N) if

(1) $\mathbf{K}\,\xi \in T$ implies $(I, \langle M, N\rangle, \langle M, N\rangle)(\mathbf{K}\,\xi) = \mathbf{t}$,

(2) $\mathbf{K}\,\xi \in F$ implies $(I, \langle M, N\rangle, \langle M, N\rangle)(\mathbf{K}\,\xi) = \mathbf{f}$, and

(3) $\mathbf{K}\,\xi \notin T$ and $\mathbf{K}\,\xi \notin F$ implies $(I, \langle M, N\rangle, \langle M, N\rangle)(\mathbf{K}\,\xi) = \mathbf{u}$.

Based on this relation, we can show that the objective knowledge derived from the partition, which is induced by a three-valued MKNF model, yields again that model.

Proposition 3. *For* $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ *a ground hybrid MKNF knowledge base, let* (M, N) *be a partial MKNF model of* $\mathcal{K}_G$*, and* (T, F) *a partition of* $\mathsf{KA}(\mathcal{K}_G)$ *induced by* (M, N)*. Then* $(M, N) = (\{I \mid I \models \mathsf{OB}_{\mathcal{O},T}\}, \{I \mid I \models \mathsf{OB}_{\mathcal{O},\mathsf{KA}(\mathcal{K}_G)\setminus F}\})$*.*

Proof. For $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ a ground hybrid MKNF knowledge base, let (M, N) be a partial MKNF model of $\mathcal{K}_G$, (T, F) a partition of $\mathsf{KA}(\mathcal{K}_G)$ induced by (M, N), and $(M', N') = (\{I \mid I \models \mathsf{OB}_{\mathcal{O},T}\}, \{I \mid I \models \mathsf{OB}_{\mathcal{O},\mathsf{KA}(\mathcal{K}_G)\setminus F}\})$. We show that $(M, N) = (M', N')$.

First, we show that $M \subseteq M'$. Let I be an interpretation in M. We show that $I \in M' = \{I \mid I \models \mathsf{OB}_{\mathcal{O},T}\}$, i.e. we show that $I \models \{\pi(\mathcal{O})\} \cup \{\xi \mid \mathbf{K}\xi \in T\}$. First, (M, N) is a partial MKNF model of $\mathcal{K}_G$, so we know that $(M, N) \models \mathbf{K}\,\pi(\mathcal{O})$. Thus, we have $I \models \pi(\mathcal{O})$. Consider in turn each $\mathbf{K}\,\xi \in T$. Since (M, N) induces a partition (T, F) we have $(I, \langle M, N\rangle, \langle M, N\rangle)(\mathbf{K}\,\xi) = \mathbf{t}$ and thus $I \models \xi$. Hence, $I \models \mathsf{OB}_{\mathcal{O},T}$. This shows that $I \in M'$ and therefore $M \subseteq M'$.

Next, we show that $N \subseteq N'$. Let I be an interpretation in N. We show that $I \in N' = \{I \mid I \models \mathsf{OB}_{\mathcal{O},\mathsf{KA}(\mathcal{K}_G)\setminus F}\}$, i.e. we show that $I \models \{\pi(\mathcal{O})\} \cup \{\xi \mid \mathbf{K}\,\xi \in \mathsf{KA}(\mathcal{K}_G) \setminus F\}$. We already know for each $I \in M$ that $I \models \pi(\mathcal{O})$. Since $N \subseteq M$ we also have that $I \models \pi(\mathcal{O})$ for each $I \in N$. Consider each $\mathbf{K}\,\xi \notin F$. This means that condition (2) of Definition 15 is not possible, conditions (1) and (3), however, are. In case of (1), we already know that $I \models \xi$ for each $I \in M$, and since $N \subseteq M$ we also have $I \models \xi$ for each $I \in N$. In case of (3), we know that $(I, \langle M, N\rangle, \langle M, N\rangle)(\mathbf{K}\,\xi) = \mathbf{t}$ for each $I \in N$. Thus, we also obtain that $I \models \xi$ holds for each $I \in N$. Then $I \models \mathsf{OB}_{\mathcal{O},\mathsf{KA}(\mathcal{K}_G)\setminus F}$ which also shows $N \subseteq N'$.

We show now that each of the two sets are in fact identical, i.e. $M = M'$ and $N = N'$. Note first that $T \subseteq \mathsf{KA}(\mathcal{K}_G) \setminus F$. Thus, for any $I \in N'$, we have that $I \in \{I \mid I \models \mathsf{OB}_{\mathcal{O},T}\}$ and therefore $N' \subseteq M'$, i.e. (M', N') is an interpretation pair. So assume that (M', N') is any interpretation pair with $M \subseteq M'$ and $N \subseteq N'$, where at least one of the inclusions is proper. We show that $(I', \langle M', N'\rangle, \langle M, N\rangle)(\pi(\mathcal{K}_G)) = \mathbf{t}$ for all $I' \in M'$ and thus derive a contradiction to (M, N) being a partial MKNF model of $\mathcal{K}_G$. The former can be shown if we can prove that $(I', \langle M', N'\rangle, \langle M, N\rangle)(\mathbf{K}\,\pi(\mathcal{O}) \wedge \pi(\mathcal{P}_G)) = \mathbf{t}$ for all $I' \in M'$. By definition of M' we know that $(I', \langle M', N'\rangle, \langle M, N\rangle)(\mathbf{K}\,\pi(\mathcal{O})) = \mathbf{t}$ for all $I' \in M'$. We only have to show the same for $\pi(\mathcal{P}_G)$. We show in a case distinction that the modal atoms appearing in $\pi(\mathcal{P}_G)$ are evaluated to identical truth values in (M, N) and (M', N'). This suffices to show that $(I', \langle M', N'\rangle, \langle M, N\rangle)(\pi(\mathcal{P}_G)) = \mathbf{t}$ for all $I' \in M'$ since (M, N) is a partial MKNF model.

- Consider each $\mathbf{K}\,\xi \in T$. We obtain $(I', \langle M', N'\rangle, \langle M, N\rangle)(\mathbf{K}\,\xi) = \mathbf{t}$ for all $I' \in M'$ by definition of M' just as we have $(I, \langle M, N\rangle, \langle M, N\rangle)(\mathbf{K}\,\xi) = \mathbf{t}$ for all $I \in M$ by Definition 15.
- Consider each $\mathbf{K}\,\xi \in F$. We obtain $(I, \langle M, N\rangle, \langle M, N\rangle)(\mathbf{K}\,\xi) = \mathbf{f}$, i.e. $(I, \langle M, N\rangle, \langle M, N\rangle)(\mathbf{K}\,\xi) = \mathbf{f}$ for some $I \in N$. Because of that and since $N \subseteq N'$ we also have $(I', \langle M', N'\rangle, \langle M, N\rangle)(\mathbf{K}\,\xi) = \mathbf{f}$ for some $I' \in N'$.
- Consider each $\mathbf{K}\,\xi$ with $\mathbf{K}\,\xi \notin F$ and $\mathbf{K}\,\xi \notin T$. By Definition 15, we obtain $(I, \langle M, N\rangle, \langle M, N\rangle)(\mathbf{K}\,\xi) = \mathbf{u}$. By definition of N' we have $(I', \langle M', N'\rangle, \langle M, N\rangle)(\mathbf{K}\,\xi) \neq \mathbf{f}$. From $(I, \langle M, N\rangle, \langle M, N\rangle)(\mathbf{K}\,\xi) = \mathbf{u}$ and $M \subseteq M'$ we conclude that only $(I', \langle M', N'\rangle, \langle M, N\rangle)(\mathbf{K}\xi) = \mathbf{u}$ is possible.

- Consider any modal **not**-atom appearing in $\pi(\mathcal{P}_G)$. Since the evaluation of these is done in both cases w.r.t. (M, N), we straightforwardly obtain the identical evaluation.

$\square$

Later, this result will be used to show that the specific partition we compute in the following yields in fact a partial MKNF model (see Theorem 4).

The following example illustrates the previously introduced notions.

Example 6. Consider $\mathcal{K}$ consisting only of the rule (8) from Example 3 and an ontology containing just one assertion:

$$\mathrm{CD}(\mathrm{BNAW}) \tag{15}$$

The ground KB $\mathcal{K}_G$ contains one rule which results from (8) by substituting x with BNAW. We thus obtain

$$\begin{aligned}\mathsf{KA}(\mathcal{K}_G) \quad = \quad & \{\mathbf{K}\, Recommend(BNAW), \mathbf{K}\, CD(BNAW), \mathbf{K}\, owns(BNAW), \\ & \mathbf{K}\, lowEval(BNAW), \mathbf{K}\, interesting(BNAW)\}.\end{aligned}$$

One can easily check that there is only one partial MKNF model (M, N) of $\mathcal{K}_G$, namely the one in which each $I \in M, N$ satisfies $\mathrm{CD}(\mathrm{BNAW}) \in I$. This partial MKNF model induces a partition in which CD(BNAW) appears in T and all other modal **K**-atoms in N. The related set of first-order formulas just contains CD(BNAW). This is reasonable since the grounded version of (8) does not allow us to derive anything from it, so we can ignore it when considering models of $\mathcal{K}_G$.

4.2. Computation of the Alternating Fixpoint

As we have seen in Section 3, there are usually several partial MKNF models. We have special interest in one particular of them, namely the outlined well-founded MKNF model which is least w.r.t. derivable knowledge, and more specifically in how to compute that model. In order to obtain it from its corresponding partial partition, we resort to several existing relations and correspondences with semantics from Logic Programming. First, stable models ([17]) for normal logic programs correspond one-to-one to two-valued MKNF models of hybrid MKNF knowledge bases containing just rules (see [37]). Like (two-valued) MKNF models, stable models are in general not iteratively computed but have to be guessed and a specific operator allows us to verify whether the guess actually corresponds to a stable model. The Well-Founded Semantics ([53]) for normal logic programs on the contrary can be computed by an iteration, though being slightly weaker w.r.t. to derivable information than the stable model semantics. In particular, the data complexity of this computation is P while the data complexity of guessing stable models is coNP, making the

Well-Founded Semantics thus less expressive but easier to compute. Interestingly, there is nevertheless a relation between the two approaches shown in [52]: the operator which checks whether stable models are indeed stable can be used to compute the Well-Founded Semantics by the so-called alternating fixpoint computation. Intuitively, an operator, which results from applying twice the stable models operator, is used to compute the least and the greatest fixpoint which correspond, respectively, to the true and non-false knowledge. The term alternating stems from the fact that the computation considering the single stable model operator is not monotonically increasing or decreasing but an alternating sequence of underestimating and overestimating intermediate results which get closer to the final result in every iteration.

Since stable models and (two-valued) MKNF models, as already said, are closely related we are going to take advantage of the scheme provided in [52] and adapt it to hybrid MKNF knowledge bases. We define operators which provide a stable condition for nondisjunctive hybrid MKNF knowledge bases and use them to obtain a fixpoint, i.e. the partition corresponding to the well-founded MKNF model.

This adaptation is not straightforward as we have to deal with two problems arising from the combination of classical negation in the ontology part and nonmonotonic negation in the rules. One problem are possible inconsistencies appearing between rules and the ontology, e.g. an atom which can be derived to be false from the ontology but derived to be true from the rules. The other problem is called *coherence problem* ([45]): a first-order false formula φ (as a consequence of the DL part) has to impose that $\mathbf{not}\,\varphi$ is false as well. We present two examples within our example scenario to show the relevance of these issues and start with inconsistency.

Example 7. Consider the hybrid MKNF knowledge base presented in Example 1 and 3 for recommending CDs. Now suppose, that the user wants to ensure that only CDs are recommended which are not expensive. Note that this is different from recommending CDs which have a discount. The ontology axiom (16) states that any expensive CD must never be recommended. In general, comparing prices requires some predicates from the maths domain, and e.g. concrete domains for the DL $\mathcal{EL}^{++}$ ([2]) or the built-in predicates for implementations related to logic programs permit that. Here however, for simplicity, we assume that this is handled internally and simply add a fact (17) saying that ByTheSea is expensive.

$$\text{Expensive} \sqsubseteq \neg\text{Recommend} \tag{16}$$

$$\mathbf{K}\,\text{Expensive}(\text{ByTheSea}) \leftarrow \tag{17}$$

Now, we can conclude that ByTheSea is recommended (from (8)) and not recommended at the same time, i.e. the knowledge base is inconsistent. In cases like this we want to be able to discover the inconsistency from the computation of the model and we will show below how this can be achieved.

We can also adapt our example scenario to present coherence.

Example 8. Consider again only the hybrid MKNF knowledge base presented in Example 1 and 3 for recommending CDs. Suppose, the user wants to stall recommendations until the evaluation is available. This can be achieved by e.g. adding the rule (18).

$$\mathbf{K}\,\mathrm{LowEval}(x) \quad \leftarrow \quad \mathbf{not}\,\mathrm{Recommend}(x) \tag{18}$$

$$\neg\mathrm{LowEval}(\mathrm{ByTheSea}) \tag{19}$$

Now, only if we explicitly add information confirming that a CD has no low evaluation then we can derive its recommendation. If e.g. (19) is available then we want to derive that ByTheSea is recommended and such a derivation is not possible in a naive adaptation of the alternating fixpoint construction known from Logic Programming. Note that the functionality desired in this example can be achieved in a more sophisticated way. However, similar examples can be found and since we want the approach to be robust in practice, coherence has to be addressed properly.

For both problems we require that classically negated derviations from the ontology interact with derivations from rules. However, classical negation is not expressible in MKNF rules and nonmonotonic negation is not expressible in the ontology, so we can not link the two directly. Thus, instead of representing the connection directly, we introduce new positive DL atoms which represent the falsity of an already existing DL atom, and another program transformation making these new modal atoms available for reasoning in the respective rules.

Definition 16. Let $\mathcal{K}$ be a hybrid MKNF knowledge base. We obtain $\mathcal{K}^+$ from $\mathcal{K}$ by adding an axiom $\neg P \sqsubseteq NP$ for every DL atom $P(t_1,\ldots,t_n)$ which occurs as head in at least one rule in $\mathcal{K}$ where NP is a new predicate not already occurring in $\mathcal{K}$. Moreover, we obtain $\mathcal{K}^*$ from $\mathcal{K}^+$ by adding $\mathbf{not}\,NP(t_1,\ldots,t_n)$ to the body of each rule with $P(t_1,\ldots,t_n)$ in the head.

The idea is to have NP available as a predicate representing that $\neg P$ (with its corresponding arguments) holds: $\mathcal{K}^+$ makes this connection explicit and $\mathcal{K}^*$ introduces a restriction on each rule with such a DL atom in the head saying intuitively that the rule can only be used to conclude the head if the (classical) negation of the head does not hold already. Note that $\mathcal{K}^+$ and $\mathcal{K}^*$ are still hybrid MKNF knowledge bases, so these transformations do not formally affect the applicability of any definition regarding $\mathcal{K}$. We thus only refer to $\mathcal{K}^+$ and $\mathcal{K}^*$ explicitly when it is necessary.

It is important to point out that the addition of the statements of the form $\neg P \sqsubseteq NP$ to any arbitrary DL does not alter the decidability of the DL in consideration, no matter whether P is a concept or a role. The reason is that the predicates NP do not appear elsewhere in the ontology so that no derivation of the ontology is affected. The only purpose of these predicates is to be usable in the rules. However, it is also important to note that the ontology resulting from such an addition is usually not any longer expressible in the DL of its origin since such statements for roles, i.e. binary predicates, are not expressible in most DL. But this is not necessarily a problem as indicated in [31] for the DL $\mathcal{EL}^+$. We only have to be careful in practice how to include these statements properly into the algorithmization.

We continue now by defining an operator $T_{\mathcal{K}_G}$ which allows us to draw conclusions from positive hybrid MKNF knowledge bases.

Definition 17. For $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ a positive ground hybrid MKNF knowledge base, $R_{\mathcal{K}_G}$, $D_{\mathcal{K}_G}$, and $T_{\mathcal{K}_G}$ are defined mapping $2^{\mathsf{KA}(\mathcal{K}_G^*)}$ to $2^{\mathsf{KA}(\mathcal{K}_G^*)}$ as follows:

$$\begin{aligned} R_{\mathcal{K}_G}(S) &= \{\mathbf{K}\, H \mid \mathcal{P}_G \text{ contains a rule of the form (1) such that } \mathbf{K}\, A_i \in S \\ &\qquad \text{for each } 1 \leq i \leq n\} \\ D_{\mathcal{K}_G}(S) &= \{\mathbf{K}\, \xi \mid \mathbf{K}\, \xi \in \mathsf{KA}(\mathcal{K}_\mathsf{G}^*) \text{ and } \mathsf{OB}_{\mathcal{O},S} \models \xi\} \\ T_{\mathcal{K}_G}(S) &= R_{\mathcal{K}_G}(S) \cup D_{\mathcal{K}_G}(S) \end{aligned}$$

The operator $R_{\mathcal{K}_G}$ derives immediate consequences from the rules in $\mathcal{K}_G$ while $D_{\mathcal{K}_G}$ yields consequences from the ontology combined with the already known information in S[12]. It is important to point out that these operators apply to both $\mathcal{K}_G^+$ and $\mathcal{K}_G^*$ and permit further modifications to the knowledge base in consideration, but in all cases the operators are defined for $2^{\mathsf{KA}(\mathcal{K}_G^*)}$ to ensure that certain technical properties hold for the construction we define below.

The operator $T_{\mathcal{K}_G}$ can be shown to be monotonic:

Proposition 4. *Let $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ be a positive ground hybrid MKNF knowledge base and $S, S' \subseteq \mathsf{KA}(\mathcal{K}_\mathsf{G}^*)$ with $S \subseteq S'$. Then $T_{\mathcal{K}_G}(S) \subseteq T_{\mathcal{K}_G}(S')$.*

Proof. Suppose that $\mathbf{K}\, H \in T_{\mathcal{K}_G}(S)$. By Definition 17, $\mathbf{K}\, H \in R_{\mathcal{K}_G}(S) \cup D_{\mathcal{K}_G}(S)$ holds, so we do a case distinction. If $\mathbf{K}\, H \in R_{\mathcal{K}_G}(S)$ then $\mathcal{P}_G$ contains a rule of the form (1) such that $\mathbf{K}\, A_i \in S$ for each $1 \leq i \leq n$. Since $S \subseteq S'$, we also have that $\mathbf{K}\, A_i \in S'$ for each $1 \leq i \leq n$ and $\mathbf{K}\, H \in T_{\mathcal{K}_G}(S')$. If $\mathbf{K}\, H \in D_{\mathcal{K}_G}(S)$ then $\mathbf{K}\, H \in \mathsf{KA}(\mathcal{K}_\mathsf{G}^*)$ and $\mathsf{OB}_{\mathcal{O},S} \models H$. By monotonicity of first-order logic and since $S \subseteq S'$, we also have $\mathsf{OB}_{\mathcal{O},S'} \models H$. We conclude that $\mathbf{K}\, H \in T_{\mathcal{K}_G}(S')$. □

Since $T_{\mathcal{K}_G}$ is monotonic it has a unique least fixpoint (by the Knaster-Tarski Theorem [51]) which we denote $T_{\mathcal{K}_G} \uparrow \omega$ in reference to the limit ordinal of natural numbers ω. It is important to note that the Knaster-Tarski Theorem in general only says that this fixpoint is reached for some ordinal which might easily be greater than ω. However, in this approach, since we do not allow function symbols nor infinite sets of rules, the iteration over a finite knowledge base (with finitely many ground rules) and thus finitely many modal atoms in $\mathsf{KA}(\mathcal{K}_\mathsf{G}^*)$ terminates definitely before ω so that the ordinal ω in this paper

[12] Note that in previous versions this operator was split into two parts. Implicitly, we only derive DL atoms from such an augmented ontology plus possibly some non-DL atoms in case equalities appear in the ontology.

merely serves as a representative for the unknown but finite number of iterations necessary. We obtain the least fixpoint as follows:

$$\begin{aligned} T_{\mathcal{K}_G} \uparrow 0 &= \emptyset \\ T_{\mathcal{K}_G} \uparrow (n+1) &= T_{\mathcal{K}_G}(T_{\mathcal{K}_G} \uparrow n) \\ T_{\mathcal{K}_G} \uparrow \omega &= \bigcup_{i \geq 0} T_{\mathcal{K}_G} \uparrow i \end{aligned}$$

Now, a transformation is defined for turning hybrid MKNF knowledge bases into positive ones, thus generalizing the application of the operator $T_{\mathcal{K}_G}$.

Definition 18. Let $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ be a ground hybrid MKNF knowledge base and let $S \subseteq \mathsf{KA}(\mathcal{K}_G^*)$. The *MKNF transform* $\mathcal{K}_G/S = (\mathcal{O}, \mathcal{P}_G/S)$ is defined as follows. $\mathcal{P}_G/S$ contains all rules

$$\mathbf{K}\, H \leftarrow \mathbf{K}\, A_1, \ldots, \mathbf{K}\, A_n$$

for which there exists a rule

$$\mathbf{K}\, H \leftarrow \mathbf{K}\, A_1, \ldots, \mathbf{K}\, A_n, \mathbf{not}\, B_1, \ldots, \mathbf{not}\, B_m$$

in $\mathcal{P}_G$ with $\mathbf{K}\, B_j \notin S$ for all $1 \leq j \leq m$.

This definition resembles the transformation known from stable models [17] of logic programs, i.e. we remove all rules which contain negated atoms contradicting the given set S, and we remove all remaining negated atoms from the other rules. This notion can be used in the spirit of [17] to define two operators for hybrid MKNF knowledge bases.

Definition 19. Let $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ be a ground hybrid MKNF knowledge base and $S \subseteq \mathsf{KA}(\mathcal{K}^*)$. We define:

$$\Gamma_{\mathcal{K}_G}(S) = T_{\mathcal{K}_G^+/S} \uparrow \omega \qquad \Gamma'_{\mathcal{K}_G}(S) = T_{\mathcal{K}_G^*/S} \uparrow \omega$$

One might wonder why e.g. the operator $\Gamma'_{\mathcal{K}_G}$ not suffices for our purposes but the following example presents the reason on an intuitive level.

Example 9. Consider again the hybrid MKNF knowledge base $\mathcal{K}$ from Example 7. We consider its ground version $\mathcal{K}_G$ and add in case of $\mathcal{K}_G^+$ the axiom (20). Moreover, for $\mathcal{K}_G^*$, we modify the rule (8) instantiated with ByTheSea (Bts) as shown in (21).

$$\begin{aligned} \neg\text{Recommend} &\sqsubseteq \text{NRecommend} && (20) \\ \mathbf{K}\,\text{Recommend(Bts)} &\leftarrow \mathbf{K}\,\text{CD(Bts)}, \mathbf{not}\,\text{owns}(Bts), \mathbf{not}\,\text{LowEval}(Bts), \\ &\qquad \mathbf{K}\,\text{interesting}(Bts), \mathbf{not}\,\text{NRecommend}(Bts). && (21) \end{aligned}$$

Now, $\Gamma'_{\mathcal{K}_G}$ enables us to derive from (20), (16), and (17) that NRecommend(Bts) holds. As a consequence, it is ensured that $\mathbf{K}$ Recommend(Bts) never holds.

However, $\Gamma'_{\mathcal{K}_G}$ is inconsistent and there is no possibility to discover that. Thus, we defined two operators $\Gamma_{\mathcal{K}_G}$ and $\Gamma'_{\mathcal{K}_G}$ that interact and permit the detection of inconsistencies. Intuitively, $\Gamma'_{\mathcal{K}_G}$ can be used to enforce for any modal atom $\mathbf{K}\,\xi$ that $\neg\xi$ implies $\mathbf{not}\,\xi$. Then, $\Gamma_{\mathcal{K}_G}$ checks whether any derivation obtained from that correspondence is not only based on that but also justified by other means. In this sense, the absence of any other means indicate an inconsistency.

Before we present how this interaction of the two operators is defined, we show that both operators $\Gamma_{\mathcal{K}_G}$ and $\Gamma'_{\mathcal{K}_G}$ are antitonic.

Lemma 3. *Let $\mathcal{K}_G$ be a ground hybrid MKNF knowledge base and $S \subseteq S' \subseteq \mathsf{KA}(\mathcal{K}^*_\mathsf{G})$. Then $\Gamma_{\mathcal{K}_G}(S') \subseteq \Gamma_{\mathcal{K}_G}(S)$ and $\Gamma'_{\mathcal{K}_G}(S') \subseteq \Gamma'_{\mathcal{K}_G}(S)$*

Proof. We show the argument for $\Gamma_{\mathcal{K}_G}$, the proof for $\Gamma'_{\mathcal{K}_G}$ is identical.

By Definition 19, we have to show that $T_{\mathcal{K}^+_G/S'} \uparrow \omega \subseteq T_{\mathcal{K}^+_G/S} \uparrow \omega$. We prove by induction on n that $T_{\mathcal{K}^+_G/S'} \uparrow n \subseteq T_{\mathcal{K}^+_G/S} \uparrow n$ holds. The base case for $n = 0$ is trivial since $\emptyset \subseteq \emptyset$. Assume that $T_{\mathcal{K}^+_G/S'} \uparrow n \subseteq T_{\mathcal{K}^+_G/S} \uparrow n$ holds and consider $\mathbf{K}\,H \in T_{\mathcal{K}^+_G/S'} \uparrow (n+1)$. Then $\mathbf{K}\,H \in T_{\mathcal{K}^+_G/S'}(T_{\mathcal{K}^+_G/S'} \uparrow n)$ and there are two cases to consider. First, $\mathcal{K}^+_G/S'$ contains a rule of the form $\mathbf{K}\,H \leftarrow \mathbf{K}\,A_1 \wedge \ldots \wedge \mathbf{K}\,A_n$ such that $\mathbf{K}\,A_i \in T_{\mathcal{K}^+_G/S'} \uparrow n$ for each $1 \leq i \leq n$. Since $S \subseteq S'$ we also have $\mathbf{K}\,H \subset \mathbf{K}\,A_1 \wedge \ldots \wedge \mathbf{K}\,A_n$ in $\mathcal{K}^+_G/S$ and by the induction hypothesis $\mathbf{K}\,A_i \in T_{\mathcal{K}^+_G/S} \uparrow n$ for each $1 \leq i \leq n$. Hence, $\mathbf{K}\,H \in T_{\mathcal{K}^+_G/S} \uparrow (n+1)$. Alternatively, $\mathbf{K}\,H$ is a consequence obtained from $D_{\mathcal{K}^+_G/S'}(T_{\mathcal{K}^+_G/S'} \uparrow n)$. By the induction hypothesis, $T_{\mathcal{K}^+_G/S'} \uparrow n \subseteq T_{\mathcal{K}^+_G/S} \uparrow n$ holds and we conclude from the monotonicity of first-order logic that $\mathbf{K}\,H \in D_{\mathcal{K}^+_G/S}(T_{\mathcal{K}^+_G/S} \uparrow n)$. □

Since both operators are antitonic, we can define an alternating iteration for the two operators in the manner presented in [24] for the alternating fixpoint of normal logic programs.

Definition 20. Let $\mathcal{K}_G$ be a ground hybrid MKNF knowledge base. We define two sequences $\mathbf{P}_i$ and $\mathbf{N}_i$ as follows.

$$\begin{aligned} \mathbf{P}_0 &= \emptyset & \mathbf{N}_0 &= \mathsf{KA}(\mathcal{K}^*_G) \\ \mathbf{P}_{n+1} &= \Gamma_{\mathcal{K}_G}(\mathbf{N}_n) & \mathbf{N}_{n+1} &= \Gamma'_{\mathcal{K}_G}(\mathbf{P}_n) \\ \mathbf{P}_\omega &= \bigcup \mathbf{P}_i & \mathbf{N}_\omega &= \bigcap \mathbf{N}_i \end{aligned}$$

The sequence of $\mathbf{P}_i$ is intended to compute modal atoms which are true, while the sequence $\mathbf{N}_i$ computes modal atoms which are not false.

We can show that the sequence of $\mathbf{P}_i$, resp. $\mathbf{N}_i$, is increasing, resp. decreasing.

Lemma 4. *Let $\mathcal{K}_G$ be a ground hybrid MKNF knowledge base. Then $\mathbf{P}_\alpha \subseteq \mathbf{P}_\beta$ and $\mathbf{N}_\beta \subseteq \mathbf{N}_\alpha$ for all ordinals α, β with $\alpha \leq \beta \leq \omega$.*

Proof. Whenever $\alpha = \beta$ then the statement holds automatically. We can thus limit α to be a successor ordinal and show via induction over α that the statement holds. If β is a successor ordinal then it is sufficient to show the property for $\beta = \alpha + 1$, all the other successor cases follow from that one by transitivity of $\subseteq$.

If $\alpha = 0$ then $\mathbf{P}_0 = \emptyset$ and $\mathbf{P}_0 \subseteq \mathbf{P}_\beta$ holds for arbitrary β. Equivalently, $\mathbf{N}_0 = \mathsf{KA}(\mathcal{K}_G^*)$, thus $\mathbf{N}_\beta \subseteq \mathbf{N}_0$ also holds for any β.

Suppose the property holds for all $\alpha \leq n$, we have to show that $\mathbf{P}_{n+1} \subseteq \mathbf{P}_{n+2}$ and $\mathbf{N}_{n+2} \subseteq \mathbf{N}_{n+1}$. We have $\mathbf{P}_{n+1} = \Gamma(\mathbf{N}_n)$ and $\mathbf{P}_{n+2} = \Gamma(\mathbf{N}_{n+1})$. Since $\mathbf{N}_{n+1} \subseteq \mathbf{N}_n$ by the induction hypothesis, we obtain by antitonicity of Γ that $\mathbf{P}_{n+1} \subseteq \mathbf{P}_{n+2}$. Likewise, we know that $\mathbf{N}_{n+1} = \Gamma'(\mathbf{P}_n)$ and $\mathbf{N}_{n+2} = \Gamma'(\mathbf{P}_{n+1})$. Since $\mathbf{P}_n \subseteq \mathbf{P}_{n+1}$ by the induction hypothesis, we obtain by antitonicity of Γ' that $\mathbf{N}_{n+2} \subseteq \mathbf{N}_{n+1}$.

The only case left is the one where $\beta = \omega$. But this case holds easily by definition: $\mathbf{P}_\alpha \subseteq \bigcup \mathbf{P}_i$ and $\bigcap \mathbf{N}_i \subseteq \mathbf{N}_\alpha$ holds for arbitrary $\alpha < \beta$. □

Like in case of $T_\mathcal{K}$, and for the very same reasons, these iterations are finite and have a least fixpoint in case of $\mathbf{P}_i$ and a greatest fixpoint in case of $\mathbf{N}_i$, and both are obtained before ω. In fact, we can show that these two fixpoints exist.

Proposition 5. *Let $\mathcal{K}_G$ be a ground hybrid MKNF knowledge base. Then $\mathbf{P}_\omega$ is the least fixpoint of the sequence of $\mathbf{P}_i$ and $\mathbf{N}_\omega$ is the greatest fixpoint of the sequence of $\mathbf{P}_i$.*

Proof. We show the argument for $\mathbf{P}_\omega$, the argument for $\mathbf{N}_\omega$ follows analogously.

We define an operator $\Phi(S) = \Gamma_{\mathcal{K}_G}(\Gamma'_{\mathcal{K}_G}(S))$ on sets $S \subseteq \mathsf{KA}(\mathcal{K}_\mathsf{G}^*)$ which is iterated as usual. It is easy to see that $\Phi \uparrow i = \mathbf{P}_{2i}$ and, thus, that Φ is monotonic. By the Knaster-Tarski Theorem we conclude that $\mathbf{P}_\omega$ is equal to the least fixpoint of the sequence of $\mathbf{P}_i$. □

This proposition also allows us to show that we can compute the least fixpoint directly from the greatest one and vice versa.

Proposition 6. *Let $\mathcal{K}_G$ be a ground hybrid MKNF knowledge base. Then $\mathbf{P}_\omega = \Gamma_{\mathcal{K}_G}(\mathbf{N}_\omega)$ and $\mathbf{N}_\omega = \Gamma'_{\mathcal{K}_G}(\mathbf{P}_\omega)$.*

Proof. We show the case of $\mathbf{N}_\omega = \Gamma'(\mathbf{P}_\omega)$, the other one follows identically. By Proposition 5, we know that $\mathbf{P}_\omega$ is the least fixpoint of the sequence of $\mathbf{P}_i$. Since the ground knowledge base is finite, there is an n such that $\mathbf{P}_n = \mathbf{P}_\omega$ for which we know that $\mathbf{P}_n = \mathbf{P}_m$ for any m with $m \geq n$. Subsequently, we have $\mathbf{N}_{n+1} = \mathbf{N}_m$ for any m with $m \geq n+1$, i.e. $\mathbf{N}_{n+1} = \Gamma'_{\mathcal{K}_G}(\mathbf{P}_\omega)$ is a fixpoint of the sequence of $\mathbf{N}_i$ and thus also of $\Gamma'_{\mathcal{K}_G}$. Assume that $\mathbf{N}_{n+1}$ is not the greatest fixpoint. Then there is $\mathbf{N}_l$, $l < n+1$, with $\mathbf{N}_l = \mathbf{N}_{l+2}$ and $\mathbf{N}_l \supset \mathbf{N}_{n+1}$. Then $\mathbf{P}_{l+1}$ also equals to a fixpoint in the sequence of $\mathbf{P}_i$ with $\mathbf{P}_{l+1}$ being necessarily smaller than $\mathbf{P}_n$. This contradicts the initial assumption that $\mathbf{P}_n$ is the least fixpoint, and finishes the proof. □

Thus, we can either compute the two sequences $\mathbf{P}_i$ and $\mathbf{N}_i$ in parallel until we reach n such that $\mathbf{P}_n = \mathbf{P}_{n+1}$ and $\mathbf{N}_n = \mathbf{N}_{n+1}$ or we compute just one of the two fixpoints in the manner sketched in the proof of Proposition 5 alternating between $\Gamma_{\mathcal{K}_G}$ and $\Gamma'_{\mathcal{K}_G}$; the other one follows by one application of either $\Gamma_{\mathcal{K}_G}$ or $\Gamma'_{\mathcal{K}_G}$.

The two fixpoints can be used to define the well-founded partition which is, as we will show below, the partition which induces the well-founded MKNF model.

Definition 21. Let $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ be a consistent ground hybrid MKNF knowledge base. We define $\mathbf{P}_{\mathcal{K}_G} = \mathbf{P}_\omega \cap \mathsf{KA}(\mathcal{K}_G)$, and $\mathbf{N}_{\mathcal{K}_G} = \mathbf{N}_\omega \cap \mathsf{KA}(\mathcal{K}_G)$. Then $(T_W, F_W) = (\mathbf{P}_{\mathcal{K}_G} \cup \{\mathbf{K}\,\pi(\mathcal{O})\}, \mathsf{KA}(\mathcal{K}_G) \setminus \mathbf{N}_{\mathcal{K}_G})$ is the *well-founded partition* of $\mathcal{K}_G$.

Both $\mathbf{P}_\mathcal{K}$ and $\mathbf{N}_\mathcal{K}$, are restricted to the modal atoms occurring in $\mathcal{K}$. Thus, the auxiliary modal atoms introduced in $\mathcal{K}_G^*$ are not present in the well-founded partition. But they are not necessary there anyway since their only objective is to prevent inconsistencies and ensure coherence in the iteration. Note that we restrict the definition to consistent hybrid MKNF knowledge bases. This is reasonable since in many cases the pair (T_W, F_W) obtained for an inconsistent knowledge base would not satisfy the conditions of a partition as given in Definition 13. The only thing missing is a means for detecting inconsistencies but before we come to that, we continue the two examples related to our example scenario.

Example 10. Consider the ground hybrid MKNF knowledge base $\mathcal{K}_G$ presented in Example 9 for recommending CDs. For simplicity we ground all the rules only with Bts ignoring thus any other CDs and limit ourselves to the following set of modal atoms using appropriate abbreviations:

$$\begin{aligned} \mathsf{KA}(\mathcal{K}_G^*) &= \{\mathbf{K}\,\mathrm{Exp(Bts)}, \mathbf{K}\,\mathrm{Rec(Bts)}, \mathbf{K}\,\mathrm{NRec(Bts)}, \mathbf{K}\,\mathrm{CD(Bts)}, \\ &\quad \mathbf{K}\,\mathrm{LowEv(Bts)}, \mathbf{K}\,\mathrm{owns(Bts)}, \mathbf{K}\,\mathrm{int(Bts)}\} \end{aligned}$$

We also simplify (9) to the fact (22) and add explicitly that Bts is a CD.

$$\mathbf{K}\,\mathrm{interesting(ByTheSea)} \leftarrow . \tag{22}$$

$$\mathrm{CD(ByTheSea)} \tag{23}$$

In the following, we only consider (16) - (17) and (20) - (23). Note that **not** NRec(Bts) in (21) is not apearing in $\mathcal{K}_G^+$ and that we omit showing the transformations for $\mathcal{K}_G^*$ w.r.t. Exp(Bts) since they will not have any impact on this example.

Now we compute the two fixpoints and we start with $\mathbf{P}_0 = \emptyset$ and $\mathbf{N}_0 = \mathsf{KA}(\mathcal{K}_G^*)$. We continue with $\mathbf{P}_1 = \Gamma_{\mathcal{K}_G}(\mathbf{N}_0)$ and $\mathbf{N}_1 = \Gamma'_{\mathcal{K}_G}(\mathbf{P}_0)$. We obtain:

$$\begin{aligned} \mathbf{P}_1 &= \{\mathbf{K}\,\mathrm{CD(Bts)}, \mathbf{K}\,\mathrm{int(Bts)}, \mathbf{K}\,\mathrm{Exp(Bts)}, \mathbf{K}\,\mathrm{NRec(Bts)}\} \\ \mathbf{N}_1 &= \mathsf{KA}(\mathcal{K}_G^*) \end{aligned}$$

We have $\mathbf{K}\,\mathrm{Rec}(\mathrm{Bts}) \in \mathbf{N}_1$ but not $\mathbf{K}\,\mathrm{Rec}(\mathrm{Bts}) \in \mathbf{P}_1$ since the transform w.r.t. $\mathsf{KA}(\mathcal{K}_G^*)$ removes the rule (20) for the computation of $\mathbf{P}_1$. Note that once $\mathbf{K}\mathrm{Rec}(\mathrm{Bts})$ is derived in the computation of $\mathbf{N}_1$ and added to the set S of derived knowledge of the operator $T_{\mathcal{K}_G^*/\emptyset}$ then $D_{\mathcal{K}_G^*/\emptyset}$ permits to derive everything since $\mathsf{OB}_{\mathcal{O},S}$ is inconsistent.

We continue with $\mathbf{P}_2 = \Gamma_{\mathcal{K}_G}(\mathbf{N}_1)$ and $\mathbf{N}_2 = \Gamma'_{\mathcal{K}_G}(\mathbf{P}_1)$. We obtain:

$$\begin{aligned}
\mathbf{P}_2 &= \{\mathbf{K}\,\mathrm{CD}(\mathrm{Bts}), \mathbf{K}\,\mathrm{int}(\mathrm{Bts}), \mathbf{K}\,\mathrm{Exp}(\mathrm{Bts}), \mathbf{K}\,\mathrm{NRec}(\mathrm{Bts})\}\\
\mathbf{N}_2 &= \{\mathbf{K}\,\mathrm{CD}(\mathrm{Bts}), \mathbf{K}\,\mathrm{int}(\mathrm{Bts}), \mathbf{K}\,\mathrm{Exp}(\mathrm{Bts}), \mathbf{K}\,\mathrm{NRec}(\mathrm{Bts})\}
\end{aligned}$$

Since $\mathbf{K}\mathrm{NRec}(\mathrm{Bts}) \in \mathbf{P}_1$ holds, the rule (21) is no longer appearing in the transform used for computing $\mathbf{N}_2$, and the explosive behavior of $D_{\mathcal{K}_G^*/\emptyset}$ disappears as well. As a consequence, in the next iteration we obtain $\mathbf{K}\,\mathrm{Rec}(\mathrm{Bts}) \in \mathbf{P}_3$.

$$\begin{aligned}
\mathbf{P}_3 &= \{\mathbf{K}\,\mathrm{CD}(\mathrm{Bts}), \mathbf{K}\,\mathrm{int}(\mathrm{Bts}), \mathbf{K}\,\mathrm{Exp}(\mathrm{Bts}), \mathbf{K}\,\mathrm{NRec}(\mathrm{Bts}), \mathbf{K}\,\mathrm{Rec}(\mathrm{Bts})\}\\
\mathbf{N}_3 &= \{\mathbf{K}\,\mathrm{CD}(\mathrm{Bts}), \mathbf{K}\,\mathrm{int}(\mathrm{Bts}), \mathbf{K}\,\mathrm{Exp}(\mathrm{Bts}), \mathbf{K}\,\mathrm{NRec}(\mathrm{Bts})\}
\end{aligned}$$

These are already the fixpoints and now $\mathbf{K}\mathrm{Rec}(\mathrm{Bts}) \in \mathbf{P}_3$ but $\mathbf{K}\mathrm{Rec}(\mathrm{Bts}) \notin \mathbf{N}_3$. This is caused by the different knowledge bases $\Gamma_{\mathcal{K}_G}$ and $\Gamma'_{\mathcal{K}_G}$ are working on, and the fact that $\mathbf{K}\,\mathrm{Rec}(\mathrm{Bts})$ is intuitively true and false at the same time is already a clear indication for the inconsistency of the considered knowledge base.

While we still need to show how to discover inconsistencies formally, coherence can already be ensured.

Example 11. Consider again only the hybrid MKNF knowledge base presented in Example 8. We ground (18) and add the relevant modal atom/axiom w.r.t. $\mathcal{K}_G^*$ in (24) and (25).

$$\mathbf{K}\,\mathrm{LowEval}(\mathrm{Bts}) \leftarrow \mathbf{not}\,\mathrm{Recommend}(\mathrm{Bts}), \mathbf{not}\,\mathrm{NLowEval}(\mathrm{Bts}) \tag{24}$$
$$\neg\mathrm{LowEval} \sqsubseteq \mathrm{NLowEval} \tag{25}$$

We only consider (19) - (25) and limit ourselves to the following modal atoms:

$$\begin{aligned}
\mathsf{KA}(\mathcal{K}_G^*) = \{&\mathbf{K}\,\mathrm{Rec}(\mathrm{Bts}), \mathbf{K}\,\mathrm{LowEv}(\mathrm{Bts}), \mathbf{K}\,\mathrm{NLowEv}(\mathrm{Bts}), \mathbf{K}\,\mathrm{CD}(\mathrm{Bts}),\\
&\mathbf{K}\,\mathrm{owns}(\mathrm{Bts}), \mathbf{K}\,\mathrm{int}(\mathrm{Bts})\}
\end{aligned}$$

Here we compute the fixpoints as sketched in the proof of Proposition 6. We start with $\mathbf{P}_0 = \emptyset$ and compute $\mathbf{N}_1$ and $\mathbf{P}_2$:

$$\begin{aligned}
\mathbf{N}_1 &= \mathsf{KA}(\mathcal{K}_G^*)\\
\mathbf{P}_2 &= \{\mathbf{K}\,\mathrm{CD}(\mathrm{Bts}), \mathbf{K}\,\mathrm{int}(\mathrm{Bts}), \mathbf{K}\,\mathrm{NLowEv}(\mathrm{Bts})\}
\end{aligned}$$

Both $\mathbf{K}\,\mathrm{LowEv}(\mathrm{Bts})$ and $\mathbf{K}\,\mathrm{Rec}(\mathrm{Bts})$ are yet undefined. We continue with $\mathbf{P}_2$ and $\mathbf{N}_2$ and obtain:

$$\begin{aligned}
\mathbf{N}_3 &= \{\mathbf{K}\,\mathrm{CD}(\mathrm{Bts}), \mathbf{K}\,\mathrm{int}(\mathrm{Bts}), \mathbf{K}\,\mathrm{NLowEv}(\mathrm{Bts}), \mathbf{K}\,\mathrm{Rec}(\mathrm{Bts})\}\\
\mathbf{P}_4 &= \{\mathbf{K}\,\mathrm{CD}(\mathrm{Bts}), \mathbf{K}\,\mathrm{int}(\mathrm{Bts}), \mathbf{K}\,\mathrm{NLowEv}(\mathrm{Bts}), \mathbf{K}\,\mathrm{Rec}(\mathrm{Bts})\}
\end{aligned}$$

Now $\mathbf{K}\,\mathrm{LowEv(Bts)}$ does not occur in $\mathbf{N}_3$ since $\mathbf{K}\,\mathrm{NLowEv(Bts)}$ removes (24) from the respective transform. As a consequence we derive $\mathbf{K}\,\mathrm{Rec(Bts)}$ in $\mathbf{P}_4$. We can compute the greatest fixpoint $\mathbf{N}_\omega = \Gamma'_{\mathcal{K}_G}(\mathbf{P}_\omega)$ and obtain that it is equal to $\mathbf{N}_3$. Note that if axiom (19) is omitted then both $\mathbf{K}\,\mathrm{LowEv(Bts)}$ and $\mathbf{K}\,\mathrm{Rec(Bts)}$ remain undefined. Thus, its presence shows how the formula $\neg\mathrm{LowEv(Bts)}$ imposes that $\mathbf{not}\,\mathrm{LowEv(Bts)}$ holds, ensuring in this example the derivability of $\mathbf{K}\,\mathrm{Rec(Bts)}$.

4.3. The Well-Founded MKNF Model and Related Properties

The well-founded partition (T_W, F_W) consists of modal atoms which are intended to be true (T_W), false (F_W) or undefined (those modal atoms neither occurring in T_W nor in F_W). But this is not merely an intention. In fact, the two sequences of $\mathbf{P}_i$ and $\mathbf{N}_i$ allow us to show that any modal atom which is added to an element of the sequence of $\mathbf{P}_i$ (resp. removed from an element of the sequence of $\mathbf{N}_i$) must be true in all partial MKNF models (resp. false).

Lemma 5. *Let $\mathcal{K}_G$ be a ground hybrid MKNF knowledge base. Then $\mathbf{K}\,H \in \mathbf{P}_i$ implies that $\mathbf{K}\,H$ is true (and $\mathbf{not}\,H$ is false) in all partial MKNF models (M,N) of $\mathcal{K}_G^+$ and $\mathbf{K}\,H \notin \mathbf{N}_i$ implies that $\mathbf{K}\,H$ is false (and $\mathbf{not}\,H$ is true) in all partial MKNF models (M,N) of $\mathcal{K}_G^*$.*

Proof. We show the argument for $\mathbf{K}\,H$ by an induction on i. This also shows the argument for $\mathbf{not}\,H$ since, for all partial MKNF models (M,N) of some $\mathcal{K}$, we have that $(I, \langle M,N\rangle, \langle M,N\rangle)(\mathbf{K}\,H) = \neg(I, \langle M,N\rangle, \langle M,N\rangle)(\mathbf{not}\,H)$.

We start with the base case $i = 0$ which trivially holds since $\mathbf{P}_0$ is empty and $\mathbf{N}_0$ equals $\mathsf{KA}(\mathcal{K}_G^*)$.

(i) Suppose that the lemma holds for all $i \leq n$. We consider $i = n+1$ for two cases, namely $\mathbf{K}\,H \in \mathbf{P}_{n+1}$ and $\mathbf{K}\,H \notin \mathbf{N}_{n+1}$.

So let $\mathbf{K}\,H \in \mathbf{P}_{n+1}$. If $\mathbf{K}\,H$ already occurs in $\mathbf{P}_n$ then $\mathbf{K}\,H$ is true in all partial MKNF models (M,N) of $\mathcal{K}_G^+$ by the induction hypothesis (i). Otherwise, $\mathbf{K}\,H \in \Gamma(\mathbf{N}_n)$, i.e. $\mathbf{K}\,H \in T_{\mathcal{K}_G^+/\mathbf{N}_n} \uparrow \omega$ but $\mathbf{K}\,H \notin \mathbf{P}_n$. Since $\mathbf{K}\,H$ is introduced by $T_{\mathcal{K}_G^+/\mathbf{N}_n} \uparrow \omega$ we know that $\mathbf{K}\,H \in T_{\mathcal{K}_G^+/\mathbf{N}_n} \uparrow j$ for some j and we show by induction on j that $\mathbf{K}\,H$ is true in all MKNF models (M,N) of $\mathcal{K}_G^+$.

The base case holds trivially since $T_{\mathcal{K}_G^+/\mathbf{N}_n} \uparrow 0$ is empty.

(ii) Suppose that the claim holds for $\mathbf{K}\,H \in T_{\mathcal{K}_G^+/\mathbf{N}_n} \uparrow j$, $j \leq m$, and consider $\mathbf{K}\,H \in T_{\mathcal{K}_G^+/\mathbf{N}_n} \uparrow m+1$.

If $\mathbf{K}\,H$ already occurs in $T_{\mathcal{K}_G^+/\mathbf{N}_n} \uparrow m$ then the claim holds automatically by the induction hypothesis (ii). Otherwise, there are two cases to consider. Either there is a positive rule $\mathbf{K}\,H \leftarrow \mathbf{K}\,A_1, \ldots \mathbf{K}\,A_n$ in $\mathcal{K}_G^+/\mathbf{N}_n$ with $\mathbf{K}\,A_i \in T_{\mathcal{K}_G^+/\mathbf{N}_n} \uparrow m$ or $\mathbf{K}\,H$ is the consequence of $D_{\mathcal{K}_G^+/\mathbf{N}_n}(T_{\mathcal{K}_G^+/\mathbf{N}_n} \uparrow m)$. In the first case, by the induction hypothesis (ii), all $\mathbf{K}\,A_i$ are true in all partial MKNF models (M,N) of $\mathcal{K}_G^+$. Additionally, there is a rule $\mathbf{K}\,H \leftarrow \mathbf{K}\,A_1, \ldots \mathbf{K}\,A_n, \mathbf{not}\,B_1, \ldots, \mathbf{not}\,B_m$ in $\mathcal{K}_G^+$ and since its solely positive version occurs in $\mathcal{K}_G^+/\mathbf{N}_n$, no $\mathbf{K}\,B_j$ occurs in $\mathbf{N}_n$, and thus, by the induction hypothesis (i), all $\mathbf{K}\,B_j$ are false in all partial MKNF models (M,N) of $\mathcal{K}_G^*$. We show that all $\mathbf{K}\,B_j$ are false in all partial

MKNF models (M, N) of $\mathcal{K}_G^+$. Assume the contrary, i.e. that there is a partial MKNF model (M', N') of $\mathcal{K}_G^+$ such that at least one such $\mathbf{K}\, B_j$ is not false in it. The only difference between $\mathcal{K}_G^+$ and $\mathcal{K}_G^*$ are the auxiliary predicates appearing in the bodies of some rules in $\mathcal{K}_G^*$, i.e. the only reason for $\mathbf{K}\, B_j$ not to be false in a partial MKNF model of $\mathcal{K}_G^+$ is the absence of $\mathbf{not}\, NB_j$ in all rules with head $\mathbf{K}\, B_j$. But if $\mathbf{K}\, NB_j$ holds for all partial MKNF models of $\mathcal{K}_G^*$ then $\mathbf{K}\, NB_j$ also holds for all partial MKNF models of $\mathcal{K}_G^+$ and so $\neg B_j$ must hold for all partial MKNF models of $\mathcal{K}_G^+$ (as this is the only way of deriving $\mathbf{K}\, NB_j$). Thus, $B_j \notin I$ for each $I \in M'$ and, since $N' \subseteq M'$, also $B_j \notin I$ for each $I \in N'$. We derive that $\mathbf{K}\, B_j$ is false in (M', N'), i.e. a contradiction from which we conclude that all $\mathbf{K}\, B_j$ are false in all partial MKNF models (M, N) of $\mathcal{K}_G^+$. Consequently, $\mathbf{K}\, H$ has to be true in all partial MKNF models (M, N) of $\mathcal{K}_G^+$. In the second case, $\mathsf{OB}_{\mathcal{O}, T_{\mathcal{K}_G^+/\mathbf{N}_n} \uparrow m} \models H$ holds. Since $\mathcal{O}$ and all modal atoms occurring in $T_{\mathcal{K}_G^+/\mathbf{N}_n} \uparrow m$ are true in all partial MKNF models of $\mathcal{K}_G^+$ (by the induction hypothesis (i)), we can immediately conclude that $\mathbf{K}\, H$ also has to be true in all partial MKNF models (M, N) of $\mathcal{K}_G^+$.

Alternatively, let $\mathbf{K}\, H \notin \mathbf{N}_{n+1}$. If already $\mathbf{K}\, H \notin \mathbf{N}_n$ then $\mathbf{K}\, H$ is false in all partial MKNF models of $\mathcal{K}_G^*$ by the induction hypothesis (i). Otherwise we have $\mathbf{K}\, H \in \mathbf{N}_n$ but $\mathbf{K}\, H \notin \mathbf{N}_{n+1}$, i.e. $\mathbf{K}\, H \in \Gamma'(\mathbf{P}_{n-1})$ but $\mathbf{K}\, H \notin \Gamma'(\mathbf{P}_n)$ and thus $\mathbf{K}\, H \in T_{\mathcal{K}_G^*/\mathbf{P}_{n-1}} \uparrow \omega$ but $\mathbf{K}\, H \notin T_{\mathcal{K}_G^*/\mathbf{P}_n} \uparrow \omega$. The removal of $\mathbf{K}\, H$ is thus caused by the usage of $\mathbf{P}_n$ for creating the transform since $\mathbf{P}_{n-1} \subseteq \mathbf{P}_n$ means that $\mathcal{K}_G^*/\mathbf{P}_n$ contains fewer rules than $\mathcal{K}_G^*/\mathbf{P}_{n-1}$. Such a removed rule with $\mathbf{K}\, H$ in the head has at least one $\mathbf{not}\, B_j$ in the body such that $\mathbf{K}\, B_j$ occurs in $\mathbf{P}_n$. By the induction hypothesis (i), $\mathbf{K}\, B_j$ is true in all partial MKNF models (M, N) of $\mathcal{K}_G^+$. Assume that there is a partial MKNF model (M', N') of $\mathcal{K}_G^*$ such that at least one $\mathbf{K}\, B_j$ is not true in (M', N'). Again, the only difference between $\mathcal{K}_G^+$ and $\mathcal{K}_G^*$ are the auxiliary predicates appearing in the bodies of some rules in $\mathcal{K}_G^*$, i.e. the only reason for $\mathbf{K}\, B_j$ not to be true in (M', N') is the presence of $\mathbf{not}\, NB_j$ in all rules with head $\mathbf{K}\, B_j$ and that $\mathbf{K}\, NB_j$ is not false in (M', N'). From that we derive that $NB_j \in I$ holds for all $I \in N'$.

But if $\mathbf{K}\, NB_j$ holds for all partial MKNF models of $\mathcal{K}_G^*$ then $\mathbf{K}\, NB_j$ also holds for all partial MKNF models of $\mathcal{K}_G^+$ and so $\neg B_j$ must hold for all partial MKNF models of $\mathcal{K}_G^+$ (as this is the only way of deriving $\mathbf{K}\, NB_j$). Thus, $B_j \notin I$ for each $I \in M'$ and, since $N' \subseteq M'$, also $B_j \notin I$ for each $I \in N'$. We derive that $\mathbf{K}\, B_j$ is false in (M', N'), i.e. a contradiction from which we conclude that all $\mathbf{K}\, B_j$ are false in all partial MKNF models (M, N) of $\mathcal{K}_G^+$.

This holds for any rule with head $\mathbf{K}\, H$ and all other removed rules which might indirectly affect the derivability of $\mathbf{K}\, H$ and we conclude that $\mathbf{K}\, H$ must be false in all partial MKNF models (M, N) of $\mathcal{K}_G^*$ by minimality construction of the MKNF semantics. □

Then, the following corollary is straightforward.

Corollary 1. *Let $\mathcal{K}_G$ be a ground hybrid MKNF knowledge base and (T, F) the pair $(\mathbf{P}_\omega, \mathsf{KA}(\mathcal{K}_G^*) \setminus \mathbf{N}_\omega)$. Then $\mathbf{K}\, H \in T$ implies that $\mathbf{K}\, H$ is true (and* $\mathbf{not}\, H$

is false) in all MKNF models (M, N) of $\mathcal{K}$ and $\mathbf{K}\,H \in F$ implies that $\mathbf{K}\,H$ is false (and **not** *H is true) in all MKNF models (M, N) of $\mathcal{K}$.*

Proof. The corollary follows immediately from Lemma 5 and Proposition 5. □

Note that, for consistent KB $\mathcal{K}_G$, the pair (T, F) in the above corollary corresponds to the well-founded partition (ignoring the auxiliary predicates occurring in $\mathcal{K}_G^*$). Thus, whenever $\mathcal{K}_G$ has a partial MKNF model, i.e. is MKNF-consistent, then the well-founded partition provides a set of modal atoms which are necessarily true, respectively false, in that model. We can now use this statement in a specific way to check for consistency itself.

Proposition 7. *Let $\mathcal{K}_G$ be a ground hybrid MKNF knowledge base, $\mathbf{P}_\omega$ the fixpoint of the sequence $\mathbf{P}_i$, and $\mathbf{N}_\omega$ be the fixpoint of the sequence $\mathbf{N}_i$. If $\Gamma'(\mathbf{P}_\omega) \subset \Gamma(\mathbf{P}_\omega)$ or $\Gamma'(\mathbf{N}_\omega) \subset \Gamma(\mathbf{N}_\omega)$ then $\mathcal{K}_G$ is MKNF inconsistent.*

Proof. We show the proof for $\mathbf{N}_\omega$, the other case follows analogously. From Proposition 6 we know that $\Gamma(\mathbf{N}_\omega) = \mathbf{P}_\omega$. Furthermore, by Corollary 1, we have that all modal atoms $\mathbf{K}\,H \in \mathbf{P}_\omega$ are true in all partial MKNF models (M, N) of $\mathcal{K}_G$. If $\Gamma'(\mathbf{N}_\omega) \subset \Gamma(\mathbf{N}_\omega)$ then there is at least one $\mathbf{K}\,H$ such that $\mathbf{K}\,H \in \Gamma(\mathbf{N}_\omega) \setminus \Gamma'(\mathbf{N}_\omega)$. The only difference between these two operators is that in case of Γ' all rules with a DL-atom $\mathbf{K}\,P$ in the head contain additionally **not** NP in the body. Thus the only reason for $\mathbf{K}\,H$ not to occur in $\Gamma'(\mathbf{N}_\omega)$ is that H is such a DL-atom (or is indirectly derived from another such DL-atom, but without loss of generality we can ignore that case) and that $\neg H$ actually is a consequence obtained from $\mathcal{O}$ via $D_{\mathcal{K}_G}$ so that $\mathbf{K}\,NH$ removes the rule with head $\mathbf{K}\,H$ in case of Γ' from the respective transform. We conclude that $\mathbf{K}\,H$ has to be true and false at the same time and, thus, that the KB $\mathcal{K}_G$ is inconsistent. □

Unfortunately neither of the two tests alone is sufficient to discover inconsistencies:

Example 12. The following knowledge base is inconsistent as $\mathbf{K}\,P(a)$ is undefined and false at the same time but only the test using $\mathbf{N}_\omega$ discovers that.

$$\begin{array}{rcl} R & \sqsubseteq & \neg P \\ \mathbf{K}\,R(A) & \leftarrow & \\ \mathbf{K}\,P(A) & \leftarrow & \mathbf{not}\,P(A) \end{array}$$

For $\mathsf{KA}(\mathcal{K}_G^*) = \{\mathbf{K}\,R(A), \mathbf{K}\,P(A), \mathbf{K}\,NP(A)\}$ we obtain $\mathbf{P}_\omega = \mathsf{KA}(\mathcal{K}_G^*)$ and $\mathbf{N}_\omega = \{\mathbf{K}\,R(A), \mathbf{K}\,NP(A)\}$ and thus $\Gamma'(\mathbf{P}_\omega) = \Gamma(\mathbf{P}_\omega) = \{\mathbf{K}\,R(A), \mathbf{K}\,NP(A)\}$ and $\Gamma'(\mathbf{N}_\omega) = \{\mathbf{K}\,R(A), \mathbf{K}\,NP(A)\} \subset \Gamma(\mathbf{N}_\omega) = \mathsf{KA}(\mathcal{K}_G^*)$.

On the other hand the following knowledge base is also inconsistent but only the test with $\mathbf{P}_\omega$ allows us to discover this.

$$\begin{array}{rcl} R & \sqsubseteq & \neg P \\ \mathbf{K}\, R(A) & \leftarrow & \\ \mathbf{K}\, P(A) & \leftarrow & \mathbf{not}\, u \\ \mathbf{K}\, u & \leftarrow & \mathbf{not}\, u \end{array}$$

For $\mathsf{KA}(\mathcal{K}_G^*) = \{\mathbf{K}\, R(A), \mathbf{K}\, P(A), \mathbf{K}\, NP(A), \mathbf{K}\, u\}$ we obtain $\mathbf{P}_\omega = \{\mathbf{K}\, R(A), \mathbf{K}\, NP(A)\}$ and $\mathbf{N}_\omega = \{\mathbf{K}\, R(A), \mathbf{K}\, NP(A), \mathbf{K}\, u\}$ and thus $\Gamma'(\mathbf{P}_\omega) = \mathbf{N}_\omega \subset \Gamma(\mathbf{P}_\omega) = \mathsf{KA}(\mathcal{K}_G^*)$ and $\Gamma'(\mathbf{N}_\omega) = \Gamma(\mathbf{N}_\omega) = \mathbf{P}_\omega$.

This example is particularly interesting as it shows that neither looking for $\mathbf{K}\, H$ such that $\mathbf{K}\, H$ and $\mathbf{K}\, NH$ appear simultaneously in $\mathbf{P}_\omega$ or $\mathbf{N}_\omega$ nor comparing $\mathbf{P}_\omega$ and $\mathbf{N}_\omega$ is sufficient: both contain $\mathbf{K}\, NP(A)$ and $\mathbf{N}_\omega$ contains additionally $\mathbf{K}\, u$, i.e. $\mathbf{K}\, P(A)$ is false and $P(A)$ first-order false, however, the rule with $\mathbf{K}\, P(A)$ in the head has an undefined body and is thus not satisfied.

The conditions given in Proposition 7 alone do not suffice either to detect all inconsistencies in $\mathcal{K}_G$, since an inconsistent ontology $\mathcal{O}$ is not detected by this method. In fact, in case we want to check for consistency of $\mathcal{K}_G$ we have to check consistency of $\mathcal{O}$ alone and then apply the proposition above. Only if both results are positive then the knowledge base $\mathcal{K}_G$ is MKNF-consistent.

Theorem 2. *Let $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ be a ground hybrid MKNF knowledge base, $\mathbf{P}_\omega$ the fixpoint of the sequence $\mathbf{P}_i$, and $\mathbf{N}_\omega$ be the fixpoint of the sequence $\mathbf{N}_i$. $\mathcal{K}_G$ is MKNF-inconsistent iff $\Gamma'(\mathbf{P}_\omega) \subset \Gamma(\mathbf{P}_\omega)$ or $\Gamma'(\mathbf{N}_\omega) \subset \Gamma(\mathbf{N}_\omega)$ or $\mathcal{O}$ is inconsistent.*

Proof. One direction of the proof is a direct consequence of Proposition 7 and the definition of evaluation of MKNF formulas: if $\mathcal{O}$ is inconsistent then there is no first-order model of $\mathcal{O}$. Assume that (M, N) is a partial MKNF model of $\mathcal{K}_G$. Then, (M, N) satisfies $\mathcal{K}_G$ and thus also $\mathcal{O}$, i.e. $(I, \langle M, N\rangle, \langle M, N\rangle)(\pi(\mathcal{O})) = \mathbf{t}$ for each $I \in M$. Since M must not be empty, we derive a contradiction.

For the other direction, we have to show that any possibly occurring MKNF-inconsistency is detected. So suppose that $\mathcal{K}_G$ is MKNF-inconsistent. If $\mathcal{O}$ is inconsistent then we are done immediately. Otherwise, the rules in $\mathcal{P}_G$ alone cannot be MKNF-inconsistent since they only consist of modal atoms without any appearence of classical negation. Likewise, rules without DL-atoms or rules without DL-atoms in at least some head cannot be inconsistent since the derivation from the ontology $\mathcal{O}$ never conflicts with any rule. Consider thus such an arbitrary DL-atom $\mathbf{K}\, H$ with a rule $\mathbf{K}\, H \leftarrow \mathbf{K}\, A_1, \ldots, \mathbf{K}\, A_n, \mathbf{not}\, B_1, \ldots, \mathbf{not}\, B_m$ in $\mathcal{P}_G$. If H is true as a consequence of $\mathcal{O}$ then the operator $D_{\mathcal{K}_G}$ ensures that $\mathbf{K}\, H$ is true as well and no inconsistency occurs.

So let H be first-order false and $\mathbf{K}\, H \in \mathbf{P}_\omega$, i.e. $\mathbf{K}\, H$ is true in all partial MKNF models. But then $\Gamma'(\mathbf{N}_\omega) \subset \Gamma(\mathbf{N}_\omega)$ and the inconsistency is detected.

Alternatively, $\mathbf{K}\, H$ could be undefined but then $\mathbf{K}\, H \in \mathbf{N}_\omega$ and this is not possible since H is first-order false and Γ' suppresses $\mathbf{K}\, H$. So the only case missing is the one where $\mathbf{K}\,H$ is false in all MKNF models as enforced by the operator Γ' but the body of at least one rule with head $\mathbf{K}\,H$ is undefined. Then $\Gamma'(\mathbf{P}_\omega) \subset \Gamma(\mathbf{P}_\omega)$ and the inconsistency is detected. □

As already said, normal rules alone cannot be inconsistent, only if we allow integrity constraints which are rules whose head is $\mathbf{K}\,\mathbf{f}$ (cf. [43]). But then inconsistencies are easily detected since $\mathbf{K}\,\mathbf{f}$ must occur in $\mathsf{KA}(\mathcal{K}_G^*) \setminus \mathbf{N}_\omega$.

Example 13. Reconsider $\mathcal{K}_G$ from Example 10. We have $\mathbf{P}_\omega = \mathbf{P}_3$ and $\mathbf{N}_\omega = \mathbf{N}_3$. We check for inconsistency (assuming that the consistency check for $\mathcal{O}$ alone succeeded) and obtain $\Gamma'_{\mathcal{K}_G}(\mathbf{P}_\omega) \subset \Gamma_{\mathcal{K}_G}(\mathbf{P}_\omega)$ and $\Gamma'_{\mathcal{K}_G}(\mathbf{N}_\omega) \subset \Gamma_{\mathcal{K}_G}(\mathbf{N}_\omega)$. We conclude that $\mathcal{K}_G$ is inconsistent and does not have a well-founded partition.

Now reconsider $\mathcal{K}_G$ from Example 11. We have $\mathbf{P}_\omega = \mathbf{P}_4$ and $\mathbf{N}_\omega = \mathbf{N}_3$. We check for consistency and obtain $\Gamma'_{\mathcal{K}_G}(\mathbf{P}_\omega) = \Gamma_{\mathcal{K}_G}(\mathbf{P}_\omega)$ and $\Gamma'_{\mathcal{K}_G}(\mathbf{N}_\omega) = \Gamma_{\mathcal{K}_G}(\mathbf{N}_\omega)$. Hence we obtain the well-founded partition $(T_W, F_W) =$
$(\{\mathbf{K}\,\mathrm{CD}(\mathrm{Bts}), \mathbf{K}\,\mathrm{int}(\mathrm{Bts}), \mathbf{K}\,\mathrm{Rec}(\mathrm{Bts})\}, \{\mathbf{K}\,\mathrm{owns}(\mathrm{Bts}), \mathbf{K}\,\mathrm{LowEv}(\mathrm{Bts})\})$.

If $\mathcal{K}_G$ is consistent then the well-founded partition exists and it yields an interpretation pair which satisfies $\mathcal{K}_G$.

Theorem 3. *Let $\mathcal{K}_G$ be a consistent ground hybrid MKNF KB and $(T_W, F_W) = (\mathbf{P}_{\mathcal{K}_G} \cup \{\mathbf{K}\,\pi(\mathcal{O})\}, \mathsf{KA}(\mathcal{K}_G) \setminus \mathbf{N}_{\mathcal{K}_G})$ the well-founded partition of $\mathcal{K}_G$. Then $(I_P, I_N) \models \pi(\mathcal{K}_G)$ where $I_P = \{I \mid I \models \mathsf{OB}_{\mathcal{O},\mathbf{P}_{\mathcal{K}_G}}\}$ and $I_N = \{I \mid I \models \mathsf{OB}_{\mathcal{O},\mathbf{N}_{\mathcal{K}_G}}\}$.*

Proof. First of all, (I_P, I_N) is defined properly to be an interpretation pair, i.e. since any $I \in I_N$ also satisfies $\mathsf{OB}_{\mathcal{O},\mathbf{P}_{\mathcal{K}_G}}$ we obtain $I_N \subseteq I_P$. By Definition 3 we know that $\pi(\mathcal{K}_G) = \mathbf{K}\,\pi(\mathcal{O}) \wedge \pi(\mathcal{P}_G)$. Since $\pi(\mathcal{O})$ occurs in $\mathsf{OB}_{\mathcal{O},\mathbf{P}_{\mathcal{K}_G}}$ and all $I \in I_P$ model $\mathsf{OB}_{\mathcal{O},\mathbf{P}_{\mathcal{K}_G}}$, we have $(I, \langle I_P, I_N\rangle, \langle I_P, I_N\rangle)(\mathbf{K}\,\pi(\mathcal{O})) = \mathbf{t}$ for all $I \in I_P$. Thus, we only have to consider the evaluation of $(I, \langle I_P, I_N\rangle, \langle I_P, I_N\rangle)(\pi(\mathcal{P}_G))$.

We start by evaluating the modal atoms occurring in $\pi(\mathcal{P}_G)$. Let $\mathbf{K}\,H \in \pi(\mathcal{P}_G)$. Suppose at first that $\mathbf{K}\,H \in T_W$ then $(I, \langle I_P, I_N\rangle, \langle I_P, I_N\rangle)(\mathbf{K}\,H) = \mathbf{t}$. Alternatively, suppose $\mathbf{K}\,H \in F_W$. Assume that $\mathsf{OB}_{\mathcal{O},\mathbf{N}_{\mathcal{K}_G}} \models H$. In this case, $\mathbf{K}\,H \in \mathbf{N}_{\mathcal{K}_G}$ by means of $\mathcal{D}_{\mathcal{K}_G}$ and we conclude that $\mathsf{OB}_{\mathcal{O},\mathbf{N}_{\mathcal{K}_G}} \not\models H$. Therefore, we have $(I, \langle I_P, I_N\rangle, \langle I_P, I_N\rangle)(\mathbf{K}\,H) = \mathbf{f}$. Finally, let $\mathbf{K}\,H$ neither occur in T_W nor in F_W but in $\mathbf{N}_{\mathcal{K}_G}$. We know that $\mathsf{OB}_{\mathcal{O},\mathbf{N}_{\mathcal{K}_G}} \models H$ and assume $\mathsf{OB}_{\mathcal{O},\mathbf{P}_{\mathcal{K}_G}} \models H$. In this case, $\mathbf{K}\,H \in \mathbf{P}_{\mathcal{K}_G}$ by means of $\mathcal{D}_{\mathcal{K}_G}$ and we conclude that $\mathsf{OB}_{\mathcal{O},\mathbf{P}_{\mathcal{K}_G}} \not\models H$. Therefore, we have $(I, \langle I_P, I_N\rangle, \langle I_P, I_N\rangle)(\mathbf{K}\,H) = \mathbf{u}$. The cases for $\mathbf{not}\,H \in \pi(\mathcal{P})$ follow analogously, i.e. if $\mathbf{K}\,H \in T_W$ then $(I, \langle I_P, I_N\rangle, \langle I_P, I_N\rangle)(\mathbf{not}\,H) = \mathbf{f}$, if $\mathbf{K}\,H \in T_W$ then $(I, \langle I_P, I_N\rangle, \langle I_P, I_N\rangle)(\mathbf{not}\,H) = \mathbf{t}$, and otherwise we have $(I, \langle I_P, I_N\rangle, \langle I_P, I_N\rangle)(\mathbf{not}\,H) = \mathbf{u}$.

Now, we consider $\pi(\mathcal{P}_G)$ which consists of of a set of implications each corresponding to one rule in $\mathcal{P}_G$. For showing that $(I, \langle I_P, I_N\rangle, \langle I_P, I_N\rangle)(\pi(\mathcal{P}_G)) = \mathbf{t}$, we only have to guarantee that the three cases which map an implication $\supset$ to false do not occur, i.e. the cases where, in the corresponding original rule, the body is true but the head is not, respectively the body is undefined and the head is false. Assume that any of the three cases holds. If the body of such a rule is true then by the alternating fixpoint construction we have that the head is true as well contradicting these two cases. If the rule body is undefined then (by $\mathbf{N}_{\mathcal{K}_G}$ and the alternating fixpoint) we obtain that the head has to be undefined or true, again in contradiction to our assumption. Thus, $(I, \langle I_P, I_N\rangle, \langle I_P, I_N\rangle)(\pi(\mathcal{P}_G)) = \mathbf{t}$ holds. $\square$

This result can be combined with Proposition 3 to obtain that the well-founded partition results in a three-valued MKNF model.

Theorem 4. *Let $\mathcal{K}_G$ be a consistent ground hybrid MKNF KB and $(T_W, F_W) = (\mathbf{P}_{\mathcal{K}_G} \cup \{\mathbf{K}\ \pi(\mathcal{O})\}, \mathsf{KA}(\mathcal{K}_\mathsf{G}) \setminus \mathbf{N}_{\mathcal{K}_\mathsf{G}})$ the well-founded partition of $\mathcal{K}_G$. Then (I_P, I_N) where $I_P = \{I \mid I \models \mathsf{OB}_{\mathcal{O},\mathbf{P}_{\mathcal{K}_G}}\}$ and $I_N = \{I \mid I \models \mathsf{OB}_{\mathbf{N}_{\mathcal{O},\mathcal{K}_G}}\}$ is a partial MKNF model of $\mathcal{K}_G$.*

Proof. We know from Theorem 3 that (I_P, I_N) models $\pi(\mathcal{K}_G)$. By Proposition 3, this interpretation pair exactly corresponds to the one which equals to an MKNF model inducing that partition. Thus (I_P, I_N) is a partial MKNF model of $\mathcal{K}_G$. $\square$

In fact, it is not just any partial MKNF model but the well-founded MKNF model, i.e. the least one w.r.t. derivable knowledge. For that purpose, we show that the partition (T, F) induced by a partial MKNF model provides a fixpoint T for the sequence of $\mathbf{P}_i$ and a fixpoint T for the sequence of $\mathbf{P}_i$.

Lemma 6. *Let $\mathcal{K}_G$ be a consistent ground hybrid MKNF KB and (T, F) the partition induced by a partial MKNF model (M, N) of $\mathcal{K}_G$. Then T is a fixpoint of the sequence of $\mathbf{P}_i$ and F a fixpoint of the sequence of $\mathbf{N}_i$.*

Proof. We show the argument for T and the sequence of $\mathbf{P}_i$, the other case follows analogously.

The set T contains all modal $\mathbf{K}$-atoms from $\mathsf{KA}(\mathcal{K}_G)$ which are true in the partial MKNF model (M, N). We know that the sequence of $\mathbf{P}_i$ is monotonically increasing and that the operator Φ with $\Phi(S) = \Gamma_{\mathcal{K}_G}(\Gamma'_{\mathcal{K}_G}(S))$ is monotonic, i.e. $T \subseteq \Phi(T)$ holds.

We only have to show that $\Phi(T) \subseteq T$ holds as well. So we have to show that $\Gamma_{\mathcal{K}_G}(\Gamma'_{\mathcal{K}_G}(T)) \subseteq T$, i.e. that $T_{\mathcal{K}^+_G/\Gamma'_{\mathcal{K}_G}(T)} \uparrow \omega \subseteq T$ where $\Gamma'_{\mathcal{K}_G}(T) = T_{\mathcal{K}^*_G/T} \uparrow \omega$.

We show by induction on α that $T_{\mathcal{K}^+_G/\Gamma'_{\mathcal{K}_G}(T)} \uparrow \alpha \subseteq T$ holds for all α and therefore also for ω.

Let α be 0. Then $T_{\mathcal{K}^+_G/\Gamma'_{\mathcal{K}_G}(T)} \uparrow \alpha = \emptyset$ and the claim holds immediately. Suppose it holds for $\alpha = n$, we show the claim for $\alpha = n + 1$. Consider $\mathbf{K}\, H \in T_{\mathcal{K}^+_G/\Gamma'_{\mathcal{K}_G}(T)} \uparrow (n+1)$, i.e. $\mathbf{K}\, H \in T_{\mathcal{K}^+_G/\Gamma'_{\mathcal{K}_G}(T)}(T_{\mathcal{K}^+_G/\Gamma'_{\mathcal{K}_G}(T)} \uparrow n)$. At first, suppose that $\mathbf{K}\, H \in R_{\mathcal{K}^+_G/\Gamma'_{\mathcal{K}_G}(T)}(T_{\mathcal{K}^+_G/\Gamma'_{\mathcal{K}_G}(T)} \uparrow n)$. Then either $\mathbf{K}\, H \in T_{\mathcal{K}^+_G/\Gamma'_{\mathcal{K}_G}(T)} \uparrow n$ and the claim holds by induction hypothesis or $\mathcal{K}^+_G/\Gamma'_{\mathcal{K}_G}(T)$ contains a rule of the form (1) such that $\mathbf{K}\, A_i \in T_{\mathcal{K}^+_G/\Gamma'_{\mathcal{K}_G}(T)} \uparrow n$. In this case, all $\mathbf{K}\, A_i$ occur in T and are all modelled in (M, N). Thus (M, N) also models

$\mathbf{K}\, H$ and $\mathbf{K}\, H$ is induced in T, i.e. $\mathbf{K}\, H \in T$. Alternatively, suppose that $\mathbf{K}\, H \in D_{\mathcal{K}_G^+/\Gamma'_{\mathcal{K}_G}(T)}(T_{\mathcal{K}_G^+/\Gamma'_{\mathcal{K}_G}(T)} \uparrow n)$. In this case it is easy to see by induction hypothesis and by monotonicity of $D_{\mathcal{K}_G^+}$ that $\mathbf{K}\, H \in T$ as well. $\square$

From that we immediately obtain that the well-founded MKNF model is the least MKNF model w.r.t. derivable knowledge.

Theorem 5. *Let $\mathcal{K}$ be a consistent nondisjunctive DL-safe hybrid MKNF KB and let (M, N) be the partial MKNF model for $\mathcal{K}$. For any partial MKNF model (M_1, N_1) of $\mathcal{K}$ we have $(M_1, N_1) \succeq_k (M, N)$, i.e. (M, N) is the well-founded MKNF model.*

Proof. We have shown in Proposition 3 that any three-valued MKNF model induces a partition which yields the MKNF model again (via the objective knowledge). Since any partition (T, F) by Lemma 6 consists of two fixpoints, one of $\Phi_{\mathcal{K}}\ (T)$ and one of $\Psi_{\mathcal{K}}\ (F)$, and we know that the well-founded partition (T_W, F_W) is obtained from the least fixpoint of $\Phi_{\mathcal{K}}$ and the greatest fixpoint of $\Psi_{\mathcal{K}}$, we conclude that $T_W \subseteq T$ and $F_W \subseteq F$. Furthermore, we know that $M = \{I \mid I \models \mathsf{ob}_{\mathcal{K},\mathsf{T_W}}\}$ and $N = \{I \mid I \models \mathsf{ob}_{\mathcal{K},\mathsf{KA}(\mathcal{K})\setminus \mathsf{F_W}}\}$, and $M_1 = \{I \mid I \models \mathsf{ob}_{\mathcal{K},\mathsf{T}}\}$ and $N_1 = \{I \mid I \models \mathsf{ob}_{\mathcal{K},\mathsf{KA}(\mathcal{K})\setminus \mathsf{F}}\}$. It is straightforward to see that $M \supseteq M_1$ and $N_1 \supseteq N$ which by Definition 11 finishes the proof. $\square$

This central theorem not only shows that the well-founded model is in fact well-defined, since it is exactly the MKNF model which is least w.r.t. $\succeq_k$, but also that the well-founded MKNF model is a sound approximation of any total three-valued MKNF model and therefore of any two-valued MKNF model. Thus, the well-founded partition can also be used in the algorithms presented in [41] for computing a subset of that knowledge which holds in all partitions corresponding to a two-valued MKNF model.

One of the open questions in [41] was that MKNF models are not compatible with the well-founded model for logic programs. Our approach, regarding knowledge bases just consisting of rules, does coincide with the well-founded model for the corresponding (normal) logic program (though it obviously does not correspond to the stable model semantics).

Corollary 2. *Let $\mathcal{K}$ be a nondisjunctive program of MKNF rules, Π a normal logic program obtained from $\mathcal{P}$ by transforming each MKNF rule*

$$\mathbf{K}\, H \leftarrow \mathbf{K}\, A_1, \ldots, \mathbf{K}\, A_n, \mathbf{not}\, B_1, \ldots, \mathbf{not}\, B_m$$

into a clause

$$H \leftarrow A_1, \ldots, A_n, \mathbf{not}\, B_1, \ldots, \mathbf{not}\, B_m$$

of Π, let (P, N) be the well-founded MKNF model, and let W_Π be the well-founded model of Π. Then $\mathbf{K}\, H \in P$ if and only if $H \in W_\Pi$ and $\mathbf{K}\, H \in N$ if and only if $\mathbf{not}\, H \in W_\Pi$.

Finally the following theorem is obtained straightforwardly from the data complexity results for positive nondisjunctive MKNF knowledge bases in [41] where data complexity is measured in terms of A-Box assertions and rule facts.

Theorem 6. *Let $\mathcal{K}$ be a nondisjunctive DL-safe hybrid MKNF KB. Assuming that entailment of ground DL-atoms in $\mathcal{DL}$ is decidable with data complexity $\mathcal{C}$ the data complexity of computing the well-founded partition is in* $\mathrm{P}^{\mathcal{C}}$.

For comparison, the data complexity for reasoning with MKNF models in nondisjunctive programs is shown to be $\mathcal{E}^{\mathrm{P}^{\mathcal{C}}}$ where $\mathcal{E} = \mathrm{NP}$ if $\mathcal{C} \subseteq \mathrm{NP}$, and $\mathcal{E} = \mathcal{C}$ otherwise. Thus, computing the well-founded partition ends up in a strictly smaller complexity class than deriving the MKNF models. In fact, in case the description logic fragment is tractable[13], we end up with a formalism whose model is computed with a data complexity of P. This is remarkable, because, to the best of our knowledge, this is the first time that a general tractable local closed-world extension for DL's has been identified.

5. Related Work

Several proposals exist for combining rules and ontologies (see e.g. [23] for a brief survey). Basically they can be split into two groups, namely those being semantically based on first-order logics solely (such as description logics alone) and the hybrid approaches such as hybrid MKNF which provide a semantics usually combining elements from first-order logics with nonmonotonicity.

The first groups' most general approach is SWRL [27] which is the unrestricted combination of OWL-DL with function-free Horn rules, i.e. rules without negation. The approach is very expressive but undecidable, yet nevertheless generalizes many approaches in this group. Applying e.g. DL-safety to SWRL rules yields DL-safe rules [42], a decidable subset of SWRL. Notable among the approaches which are also generalized by SWRL are $\mathcal{AL}$-log [9] a combination of DL-safe positive rules and $\mathcal{ALC}$ and [36] for the system CARIN. In both cases the ontology only serves as input to the rules and not vice-versa. Besides that, Description Logic Programs (DLP) [21] are a fragment of OWL that can be transformed into logic programs of positive rules. In the same spirit as DLP, Horn-$\mathcal{SHIQ}$ [29] is a fragment of OWL which can be translated into Datalog, i.e. positive rules, and is just like DLP of tractable data complexity. Besides that, recently DLP has been generalized in [34] by Description Logic Rules, i.e. rules which allow description logics expressions within themselves, which allow to add more sophisticated constructs only available to more expressive description logics to enrich the DL on which the description logic rules are based without raising the complexity. In similar spirit is the language ELP [35], which is a polynomial language covering important parts of OWL 2, but which also allows us to model with axioms which cannot be expressed in OWL

[13] See e.g. the OWL2 profiles at http://www.w3.org/TR/owl2-profiles/.

2. All these approaches have the advantage of fitting semantically into the original (first-order) OWL semantics which also means that existing reasoners for ontologies alone are available for use. On the other hand, none of them is allowing for expressing nonmonotonic negation which exactly prevents modeling problems like those given in the introduction.

The second group of hybrid approaches which have partially nonmonotonic semantics are more or less similar in spirit to the hybrid MKNF semantics. The approach of [12] combines ontologies and rules in a modular way, i.e. keeps both parts and their semantics separate and is thus a less strong integration. The two reasoning engines nevertheless interact bidirectionally (with some limitations in the direction of the ontology to rules) via interfaces, and the dlvhex system [13] provides an implementation for that and generalizes the approach by allowing multiple sources for external knowledge with differing semantics. This work has also been extended in various ways (including probability, uncertainty, and priorities, for references see the related work section of [12]) and it includes a related well-founded semantics [14] with quite similar computational complexity as the well-founded MKNF semantics, but again in a less tight integration with some limitations on the transfer of information from ontologies to rules. The only other well-founded semantics approach is the one called hybrid programs [11] but it only allows us to transfer information from the ontology to the rules and not the other way around and is thus strictly less expressive than the well-founded MKNF semantics. The advantage of such a restriction is, however, that, opposite to [14] and the well-founded MKNF semantics, the semantics remains compatible with the standard semantics: consider two DL-atoms B_1 and B_2 and an ontology which expresses that at least one of them is true but none is a logical consequence of the ontology. Then, given rules $p \leftarrow B_1$ and $p \leftarrow B_2$, p is obtained by [11] but not in our work nor in the one by Eiter et. al. [12]. There are however several further approaches based on stable models such as [7] which use an embedding into autoepistemic logic to combine ontologies and rules tightly and which is quite similar in spirit to hybrid MKNF [41]. In fact, the embedding with epistemic rule bodies and epistemic rule heads seems to be the one most closely related, not only syntactically but also with respect to the semantic consequences. However, a precise relation to hybrid MKNF is far from being obvious since an autoepistemic interpretation in [7] is a pair of a first-order interpretation and a set of beliefs and both are not necessarily related. DL+log [49] provides a combination of rules and ontologies which separates predicates into rule and ontology predicates and evaluates the former w.r.t. answer set semantics and the latter w.r.t. a first-order semantics applying weak DL-safety, i.e. each variable in the head of a rule appears in an arbitrary positive atom in its body. Like [41], [8] generalizes [49] and several earlier related work such as [48] by Rosati within the framework of equilibrium logics. Quite similar to [49] is also [38] only that this approach does not distinguish between ontology and rules predicates. In fact, the work originated from [12] and thus from the perspective of rules but allows a much tighter integration than [12]. Yet another approach, open answer set programming [50], extends rules with open domains and add some syntactic limitations for ensuring decidability and, recently in [15], based on that an algorithm has been provided for f-hybrid knowledge bases, i.e. a combination of ontologies and rules without enforcing DL-safety but limiting predicates to tree-shapedness. A loose layering of Prolog on top of DLs, employing four-valued logic, is presented in [39].

An alternative way of introducing nonmonotonicity to ontologies is to enrich DLs with further syntactic constructs representing nonmonotonic features. Among these approaches the most closely related to our work is Description Logics of MKNF [10] which allow two modal operators in ontology axioms. An algorithm was provided in [10] for $\mathcal{ALC}$ with MKNF and has recently been improved in [30]. In [5, 18] circumscription was used for achieving nonmonotonic features and among several combinations introducing defaults to ontologies we mention [4].

6. Conclusions and Future Work

Summarizing, we have defined a well-founded semantics of (tightly integrated) hybrid KBs that is sound w.r.t. the semantics defined in [41] for MKNF KBs but has strictly lower complexity. In particular, we obtain tractability whenever the underlying description logic is tractable. To the best of our knowledge it is the first such approach for the combination of such rules and ontologies without any limitations on the transfer of information between the two. It coincides with the first-order semantics of the considered Description Logics fragment in case there are no rules, and with the well-founded semantics of normal programs in case the DL-part is empty. Moreover, we define a construction for computing the well-founded model that is also capable of detecting inconsistencies in a straightforward way.

Several lines of future research can be considered, some of which are already under investigation by us. First of all, we are working towards a general query-driven procedure capable of answering conjunctive queries under the well-founded semantics of hybrid MKNF knowledge base. In fact we have already some results in this issue: in [1] a procedure is defined, using tabled resolution, that is sound and complete w.r.t. the well-founded semantics defined here, and is terminating for several classes of knowledge bases. This procedure, which is parametric on an oracle capable of answering queries in the underlying description logic, is able to answer DL-safe conjunctive queries (i.e. conjunctive predicates with variables, where queries have to be ground when processed in the ontology) returning all correct answer substitutions for variables in the query. An implementation of this procedure, which is based on XSB Prolog[14] for the tabling resolution, is underway.

Another line of current research, is the specialisation of the semantics defined here, and corresponding procedures and implementations, for particular tractable description logics (rather then considering, in general, any decidable DL, as is done in this paper). This specific study, and implementation, for a

[14] http://xsb.sourceforge.net/

tractable description logic fragment aims at $\mathcal{EL}^{++}$ [2], and at its extension ELP [35]. We intend to provide a transformation of such specific hybrid knowledge bases into rules which then can be applied as an input to a logic programming system capable of computing the well-model of a set of rules.

Yet another topic that we are pursuing, is the definition of a paraconsistent version of the semantics defined here. It is worth noting that when inconsistencies come from the combination of the rules with the DL-part (i.e. for inconsistent KBs with a consistent DL-part), the construction still yields some results (e.g. in Example **??** we could still recommend album $A3$). This suggests that the method could be further exploited in the direction of defining a paraconsistent semantics for hybrid KBs.

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