A Coherent Well-founded Model for Hybrid MKNF Knowledge Bases

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Abstract. With the advent of the Semantic Web, the question becomes important how to best combine open-world based ontology languages, like OWL, with closed-world rules paradigms. One of the most mature proposals for this combination is known as Hybrid MKNF knowledge bases [11], which is based on an adaptation of the stable model semantics to knowledge bases consisting of ontology axioms and rules. In this paper, we propose a well-founded semantics for such knowledge bases which promises to provide better efficiency of reasoning, which is compatible both with the OWL-based semantics and the traditional well-founded semantics for logic programs, and which surpasses previous proposals for such a well-founded semantics by avoiding some issues related to inconsistency handling.

1 Introduction

The Web Ontology Language OWL[3] is a recommended standard by the W3C for modeling Semantic Web knowledge bases. It is essentially based on Description Logics (DLs) [1], and thus adheres to the open-world assumption.

It is apparent, however, and frequently being voiced by application developers, that it would be favorable to have closed-world modeling as an additional feature for ontology-based systems. This need has led to several investigations into combinations of closed-world rules paradigms with DLs, which can still be considered to be in their early stages, and the proposed solutions differ substantially.

We base our work on the claim that the integration should be as tight as possible, in the sense that conclusions from the rules affect the conclusions from the ontology and vice-versa. Among such proposals are several whose semantics is based on stable model semantics (SMS) [5] (e.g. [2, 3, 6, 11, 13]), and only few which are based on the well-founded semantics (WFS) [14], like [4, 8]. Though these WFS-based approaches are in general weaker in their derivable consequences, their faster computation (data complexity P vs. NP) should be more suitable for the intended application area, the WWW.

One of the currently most mature proposals for a tight integration is known as Hybrid MKNF knowledge bases [11], which draws on the logic of Minimal Knowledge and Negation as Failure (MKNF) [10]. [11]'s proposal evaluates knowledge bases under a stable model semantics, resulting in unfavorable computational complexities.

In this paper, we therefore define a new semantics, restricted to non-disjunctive rules, which soundly approximates the semantics of [11] and is in a strictly lower complexity class. The semantics furthermore yields the original DL-semantics when no rules are present, and the original well-founded semantics if the DL-component is empty. The semantics is furthermore coherent in the sense of [12], i.e. whenever any formula is first-order false then it is also non-monotonically false. It also allows for detecting inconsistencies between interacting ontologies and rules, and in fact does this without any substantial additional computational effort.

Due to this inconsistency handling, our proposal is superior to that of [8], which also attempted to define a WF semantics, but resulted in some unintuitive behavior in the presence of inconsistencies.

The paper is structured as follows. We first recall preliminaries on Hybrid MKNF knowledge bases in Section 2. We then introduce a running modeling example in Section 3 before introducing our well-founded semantics in Section 4. Section 5 is devoted to some basic properties, especially regarding consistency. In Section 6 we briefly compare with most similar approaches, and conclude. More details, including proofs, can be found in [9].

2 Preliminaries

At first we present the syntax of MKNF formulas taken from [11]. A *first-order atom* $P(t_1, \ldots, t_n)$ is an MKNF formula where P is a predicate and the t_i are function-free first-order terms. If φ is an MKNF formula then $\neg\varphi$, $\exists x : \varphi$, $\mathbf{K}\,\varphi$ and $\mathbf{not}\,\varphi$ are MKNF formulas and likewise $\varphi_1 \wedge \varphi_2$ and $\varphi_1 \leftarrow \varphi_2$ for MKNF formulas φ_1, φ_2. The symbols $\vee$, $\equiv$, and $\forall$ represent the usual boolean combinations of the previously introduced constructors. Substituting the free variables x_i in φ by terms t_i is denoted $\varphi[t_1/x_1, \ldots, t_n/x_n]$. Then, given a (first-order) formula φ, $\mathbf{K}\,\varphi$ is called a *modal* $\mathbf{K}$*-atom* and $\mathbf{not}\,\varphi$ a *modal* $\mathbf{not}$*-atom*. The signature Σ contains, apart from the constants occurring in the formulas, a countably infinite supply of constants not occurring in the formulas and the Herbrand universe of such a signature is denoted by $\triangle$. Moreover, the equality predicate $\approx$ in Σ is interpreted as an equivalence relation on $\triangle$.

As in [11], hybrid MKNF knowledge bases can contain any first-order fragment $\mathcal{DL}$ satisfying the following conditions: (i) each knowledge base $\mathcal{O} \in \mathcal{DL}$ can be translated into an equivalent formula $\pi(\mathcal{O})$ of function-free first-order logic with equality, (ii) it supports *A-Boxes*-assertions of the form $P(a_1, \ldots, a_n)$ for P a predicate and a_i constants of $\mathcal{DL}$ and (iii) satisfiability checking and instance checking (i.e. entailment of the form $\mathcal{O} \models P(a_1, \ldots, a_n)$) are decidable[4]. We now recall hybrid MKNF knowledge bases of [11].

Definition 1 *Let $\mathcal{O}$ be a DL knowledge base. A first-order function-free atom $P(t_1, \ldots, t_n)$ over Σ such that P is $\approx$ or it occurs in $\mathcal{O}$ is called a* DL-atom*; all other atoms are called* non-DL-atoms*. A (nondisjunctive) MKNF rule r has the following form where H, A_i,*

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[4] For more details on DL notation we refer to [1].

and B_i are first-order function free atoms:

$$\mathbf{K}\,H \leftarrow \mathbf{K}\,A_1, \ldots, \mathbf{K}\,A_n, \mathbf{not}\,B_1, \ldots, \mathbf{not}\,B_m \tag{1}$$

The sets $\{\mathbf{K}\,H\}$, $\{\mathbf{K}\,A_i\}$, and $\{\mathbf{not}\,B_i\}$ are called the rule head*, the* positive body*, and the* negative body*, respectively. A rule is* positive *if $m = 0$; r is a* fact *if $n = m = 0$. A program $\mathcal{P}$ is a finite set of MKNF rules. A* hybrid MKNF knowledge base *$\mathcal{K}$ is a pair $(\mathcal{O}, \mathcal{P})$.*

The semantics of such a knowledge base $\mathcal{K}$ is obtained by translating it into the MKNF formula $\pi(\mathcal{K}) = \mathbf{K}\,\pi(\mathcal{O}) \wedge \pi(\mathcal{P})$ where $\mathcal{P}$ is transformed by universally quantifying all the variables in each rule.

An MKNF rule r is *DL-safe* if every variable in r occurs in at least one non-DL-atom $\mathbf{K}\,B$ occurring in the body of r. A hybrid MKNF knowledge base $\mathcal{K}$ is *DL-safe* if all its rules are DL-safe. Given a hybrid MKNF knowledge base $\mathcal{K} = (\mathcal{O}, \mathcal{P})$, the *ground instantiation of* $\mathcal{K}$ is the KB $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ where $\mathcal{P}_G$ is obtained by replacing in each rule of $\mathcal{P}$ all variables with constants from $\mathcal{K}$ in all possible ways.

3 Example Scenario

Consider an online store selling, among other things, CDs. Due to the fact that many newly published CDs are simply compilations of already existing music, the owners decide to offer their customers a special service: whenever somebody likes the compilation of a certain artist he can search specifically for more music of that artist published on albums. The service shall however deny offering other compilations or products which are too similar to the already owned CD. Similarity can be defined in various ways but we assume for simplicity that this is handled internally, e.g. by counting the number of identical tracks, and encoded by predicate $Dif(x, y)$. The internal database is organized as a hybrid MKNF knowledge base including an ontology containing all available discs, their tracks and so on and whether they are albums or compilations. The following shall provide the considered service[5]:

$$Comp \sqsubseteq \neg Offer \tag{2}$$

$$\mathbf{K}\,Offer(x) \leftarrow \mathbf{not}\,owns(x), \mathbf{K}\,owns(y), \mathbf{K}\,Dif(x,y), \mathbf{K}\,artist(x,z), \mathbf{K}\,artist(y,z). \tag{3}$$

Given the input of CDs the customer owns, rule (3) offers an album x in case the customer does not own it, which is sufficiently different to a CD y he owns, where the artist z of x is the same as the artist of y. Additionally, (2) is a DL statement (translatable into $\forall x : Comp(x) \rightarrow \neg Offer(x)$) enforcing that any CD which is a compilation shall never be offered.

4 Three-valued Semantics

We start by defining three-valued structures which serve as a means for evaluating hybrid MKNF knowledge bases.

Definition 2 *A* three-valued (partial) MKNF structure *$(I, \mathcal{M}, \mathcal{N})$ consists of a Herbrand first-order interpretation I and two pairs $\mathcal{M} = \langle M, M_1\rangle$ and $\mathcal{N} = \langle N, N_1\rangle$ of sets of Herbrand first-order interpretations where any first-order atom which occurs in all elements in M (resp. N) also occurs in all elements of M_1 (resp. N_1). It is called* total *if $\mathcal{M} = \langle M, M\rangle$ and $\mathcal{N} = \langle N, N\rangle$.*

[5] Capital letters represent DL-atoms and objects/individuals while the other represent non-DL atoms and variables. Note that rule (3) is in fact DL-safe.

Set I is intended to interpret the first-order formulas while the pairs $\mathcal{M}$ and $\mathcal{N}$ evaluate the modal operators $\mathbf{K}$ and $\mathbf{not}$. MKNF formulas are thus interpreted with respect to the set $\{\mathbf{t}, \mathbf{u}, \mathbf{f}\}$ of truth values with $\mathbf{f} < \mathbf{u} < \mathbf{t}$ where the operator $\max$ (resp. $\min$) chooses the greatest (resp. least) element with respect to this ordering:

$$(I,\mathcal{M},\mathcal{N})(p(t_1,\ldots,t_n)) = \begin{cases} \mathbf{t} & \text{iff } p(t_1,\ldots,t_n) \in I \\ \mathbf{f} & \text{iff } p(t_1,\ldots,t_n) \notin I \end{cases}$$

$$(I,\mathcal{M},\mathcal{N})(\neg\varphi) = \begin{cases} \mathbf{t} & \text{iff } (I,\mathcal{M},\mathcal{N})(\varphi) = \mathbf{f} \\ \mathbf{u} & \text{iff } (I,\mathcal{M},\mathcal{N})(\varphi) = \mathbf{u} \\ \mathbf{f} & \text{iff } (I,\mathcal{M},\mathcal{N})(\varphi) = \mathbf{t} \end{cases}$$

$$(I,\mathcal{M},\mathcal{N})(\varphi_1 \wedge \varphi_2) = \min\{(I,\mathcal{M},\mathcal{N})(\varphi_1), (I,\mathcal{M},\mathcal{N})(\varphi_2)\}$$

$$(I,\mathcal{M},\mathcal{N})(\varphi_1 \leftarrow \varphi_2) = \begin{cases} \mathbf{t} \text{ iff } (I,\mathcal{M},\mathcal{N})(\varphi_1) \geq (I,\mathcal{M},\mathcal{N})(\varphi_2) \\ \mathbf{f} \text{ otherwise} \end{cases}$$

$$(I,\mathcal{M},\mathcal{N})(\exists x : \varphi) = \max\{(I,\mathcal{M},\mathcal{N})(\varphi[\alpha/x]) \mid \alpha \in \triangle\}$$

$$(I,\mathcal{M},\mathcal{N})(\mathbf{K}\,\varphi) = \begin{cases} \mathbf{t} \text{ iff } (J,\langle M,M_1\rangle,\mathcal{N})(\varphi) = \mathbf{t} \\ \quad\quad \text{for all } J \in M \\ \mathbf{f} \text{ iff } (J,\langle M,M_1\rangle,\mathcal{N})(\varphi) = \mathbf{f} \\ \quad\quad \text{for some } J \in M_1 \\ \mathbf{u} \text{ otherwise} \end{cases}$$

$$(I,\mathcal{M},\mathcal{N})(\mathbf{not}\,\varphi) = \begin{cases} \mathbf{t} \text{ iff } (J,\mathcal{M},\langle N,N_1\rangle)(\varphi) = \mathbf{f} \\ \quad\quad \text{for some } J \in N_1 \\ \mathbf{f} \text{ iff } (J,\mathcal{M},\langle N,N_1\rangle)(\varphi) = \mathbf{t} \\ \quad\quad \text{for all } J \in N \\ \mathbf{u} \text{ otherwise} \end{cases}$$

Note that first-order atoms, and also first-order (non-modal) formulas, are evaluated with respect to one first-order interpretation, and are, therefore, entirely two-valued. This is intended since this way a rule free hybrid knowledge base shall be interpreted just as any DL base. Moreover, implications are not interpreted in a classical sense: $\mathbf{u} \leftarrow \mathbf{u}$ is true while its classical boolean correspondence, $\mathbf{u} \vee \neg\mathbf{u}$, is undefined. This is needed for the very same reason it is in case of the well-founded semantics of LP: rules propagating undefinedness are true. Without this, it could never be the case that a DL-free hybrid knowledge would coincide with the well-founded semantics of LPs.

So, only modal atoms (and thus rules) make use of the third truth value $\mathbf{u}$ and we are going to explain the details of this part of the evaluation scheme. Each modal operator is evaluated with respect to a pair of sets of interpretations. The idea is that $\mathbf{K}\,\varphi$ is true if φ is true in all elements in M; otherwise it is either false or undefined depending on M_1. If φ is true in all elements in M_1 then $\mathbf{K}\,\varphi$ is undefined; otherwise false. The case of $\mathbf{not}\,\varphi$ is handled symmetrically with respect to $\langle N, N_1\rangle$, only now the condition for true modal $\mathbf{K}$-atoms yields false modal $\mathbf{not}$-atoms. The restrictions on three-valued MKNF structures guarantee that no modal formula can be true and false at the same time.

We now define interpretation pairs which form the basis for a model notion.

Definition 3 *An* interpretation pair *(M, N) consists of two sets of Herbrand interpretations M, N with $N \subseteq M$, and* models *a closed MKNF formula φ if and only if $(I, \langle M,N\rangle, \langle M,N\rangle)(\varphi) = \mathbf{t}$ for each $I \in M$. If there exists an interpretation pair modeling φ then φ is* consistent*.*

M contains all interpretations which model only truth while N models everything which is true or undefined. Note that the correspondence $\neg\mathbf{K}$ being equivalent to $\mathbf{not}$ is supported by using the interpretation pair (M, N) to evaluate both, $\mathbf{K}$ and $\mathbf{not}$, simultaneously. The subset relation between M and N does not only guarantee allowed MKNF structures but also that any formula which is true (resp. false), in all elements of M is also true (resp. false) in all elements of N. This will ensure consistency since it prevents modal atoms which are true and false at the same time and, as we will soon see, modal atoms which are undefined though being first-order false.

Now we define MKNF models based on a preference relation over interpretation pairs which model the considered formula.

Definition 4 *Any interpretation pair (M, N) is a* partial (or three-valued) MKNF model *for a given closed MKNF formula φ if*

(1) $(I, \langle M, N\rangle, \langle M, N\rangle)(\varphi) = \mathbf{t}$ *for all $I \in M$ and*
(2) $(I', \langle M', N'\rangle, \langle M, N\rangle)(\varphi) \neq \mathbf{t}$ *for some $I' \in M'$ and each interpretation pair (M', N') with $M \subseteq M'$ and $N \subseteq N'$ where at least one of the inclusions is proper.*

If there is a partial MKNF model of a given closed MKNF formula φ then φ is called MKNF-consistent*, otherwise* MKNF-inconsistent*.*

With a fixed evaluation of the modal $\mathbf{not}$-atoms we maximize the sets which evaluate modal $\mathbf{K}$-atoms, checking whether this still yields a true evaluation. By maximizing these sets we naturally obtain less formulas which are true in all elements of these sets and thus less modal $\mathbf{K}$-atoms which are true or undefined. In this sense we deal with a logics of minimal knowledge. Once more, by $N' \subseteq M'$, we guarantee that only reasonable augmentations are considered.

Example 1 *Consider the knowledge base from the running example, with the obvious abbreviations, together with the users input $owns(C1)$, and two albums $A1$, $A2$, in the database from the same artist, where $A1$ is sufficiently different from the compilation while $A2$ is not. Then, restricted to the domain of interest, an interpretation pair (M, N) modeling the KB and containing $owns(C1)$, $O(A1)$, and $O(A2)$ is not an MKNF model since any (M', N) such that $O(A2)$ is not in all elements of M' still models the KB. In fact, the only MKNF model restricted to these three modal atoms would be (M, N) with $M = N = \{\{owns(C1), O(A1), O(A2)\}, \{owns(C1), O(A1)\}\}$. One could ask now, what is the point of having $\mathbf{u}$ available in this example? The answer is that this simply depends on the intention and design of the reasoning capability: the idea could be only to recommend one disk. For that, we could add $\mathbf{not}\, O(x1), x \neq x1$ to the rule (3) (ensuring additionally DL-safety). Supposing that both, $A1$ and $A2$ are sufficiently different from $C1$ we would obtain two MKNF models of the KB, one with $\mathbf{K}\, O(A1)$ and one with $\mathbf{K}\, O(A2)$, and additionally one model which simply does not choose between the two but leaves them both undefined. The advantage of this comes into play when defining a way of calculating a model which incorporates all the minimally necessary true information: it is simpler to compute one slightly less expressive model than to keep track of various of them.*

Since MKNF models are in general infinite, as in [11] the proper idea for algorithmization is to represent them via a 1st-order formula whose model corresponds to the MKNF model. For that, a partition (T, F) of true and false modal atoms is provided which allows to determine this first-order formula.

Definition 5 *Let $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ be a hybrid MKNF knowledge base. The* set of $\mathbf{K}$-atoms of $\mathcal{K}$*, written $\mathsf{KA}(\mathcal{K})$, is the smallest set that contains (i) all $\mathbf{K}$-atoms occurring in $\mathcal{P}_G$, and (ii) a modal atom $\mathbf{K}\,\xi$ for each modal atom $\mathbf{not}\,\xi$ occurring in $\mathcal{P}_G$.*

For a subset S of $\mathsf{KA}(\mathcal{K})$, the objective knowledge *of S is the formula $\mathsf{ob}_{\mathcal{K},S} = \mathcal{O} \cup \bigcup_{\mathbf{K}\,\xi \in S} \xi$, and $\widehat{S}_{DL} = \{\xi \mid \mathbf{K}\,\xi \in S_{DL}\}$ where S_{DL} is the subset of DL-atoms of S. A* (partial) partition *(T, F) of $\mathsf{KA}(\mathcal{K})$ is* consistent *if $\mathsf{ob}_{\mathcal{K},T} \not\models \xi$ for each $\mathbf{K}\,\xi \in F$.*

Before we continue defining operators which will derive conclusions from knowledge bases, we have to modify MKNF knowledge bases such that we can address the coherence problem: a first-order false formula φ (as a consequence of the DL part) has to be connected to $\mathbf{not}\,\varphi$ which cannot be done straightforwardly since $\mathbf{not}$ cannot occur in the DL part. Thus, instead of representing the connection directly, we introduce new positive DL atoms which represent the falsity of an already existing DL atom, and a further program transformation which makes these new modal atoms available for reasoning in the respective rules.

Definition 6 *Let $\mathcal{K}$ be a DL-safe hybrid MKNF knowledge base. We obtain $\mathcal{K}^*$ from $\mathcal{K}$ by adding an axiom $\neg H \sqsubseteq NH$ for every DL atom $H(t_1, \ldots, t_n)$ which occurs as head in at least one rule in $\mathcal{K}$ where NH is a new predicate not allready occurring in $\mathcal{K}$. Moreover, we obtain $\widehat{\mathcal{K}}$ from $\mathcal{K}^*$ by adding $\mathbf{not}\, NH(t_1, \ldots, t_n)$ to the body of each rule with a DL atom $H(t_1, \ldots, t_n)$ in the head.*

The idea is to have $NH(t_1, \ldots, t_n)$ available as a predicate representing that $\neg H(t_1, \ldots, t_n)$ holds: $\mathcal{K}^*$ makes this connection explicit and $\widehat{\mathcal{K}}$ introduces a restriction on each rule with a DL atom in the head saying intuitively that the rule can only be used to conclude something if the negation of its head does not hold already. Note that $\mathcal{K}^*$ and $\widehat{\mathcal{K}}$ are still hybrid MKNF knowledge bases, so we only refer to $\mathcal{K}^*$ and $\widehat{\mathcal{K}}$ explicitly when it is necessary.

We now define the monotonic operator $T_{\mathcal{K}}$ which allows to draw conclusions from positive hybrid MKNF knowledge bases.

Definition 7 *For $\mathcal{K}$ a positive nondisjunctive DL-safe hybrid MKNF knowledge base, $R_{\mathcal{K}}$, $D_{\mathcal{K}}$, and $T_{\mathcal{K}}$ are defined on the subsets of $\mathsf{KA}(\widehat{\mathcal{K}})$ as follows:*

$$\begin{aligned}
R_{\mathcal{K}}(S) &= S \cup \{\mathbf{K}\, H \mid \mathcal{K} \text{ contains a rule of the form (1) such} \\
&\qquad \text{that } \mathbf{K}\, A_i \in S \text{ for each } 1 \leq i \leq n\} \\
D_{\mathcal{K}}(S) &= \{\mathbf{K}\,\xi \mid \mathbf{K}\,\xi \in \mathsf{KA}(\widehat{\mathcal{K}}) \text{ and } \mathcal{O} \cup \widehat{S}_{DL} \models \xi\} \cup \\
&\qquad \{\mathbf{K}\, Q(b_1, \ldots, b_n) \mid \mathbf{K}\, Q(a_1, \ldots, a_n) \in S \setminus S_{DL}, \\
&\qquad \mathbf{K}\, Q(b_1, \ldots, b_n) \in \mathsf{KA}(\widehat{\mathcal{K}}), \text{ and} \\
&\qquad \mathcal{O} \cup \widehat{S}_{DL} \models a_i \approx b_i \text{ for } 1 \leq i \leq n\} \\
T_{\mathcal{K}}(S) &= R_{\mathcal{K}}(S) \cup D_{\mathcal{K}}(S)
\end{aligned}$$

$R_{\mathcal{K}}$ derives immediate consequences from the rules while $D_{\mathcal{K}}$ obtains consequences using modal atoms and the statements contained in the DL part. Since $T_{\mathcal{K}}$ is monotonic, it has a unique least fixpoint which we denote $T_{\mathcal{K}} \uparrow \omega$ and is obtained in the usual way. Note that $T_{\mathcal{K}} \uparrow \omega$ is in fact order-continuous due to the absence of function symbols in the language and the restriction to known individuals.

A transformation for nondisjunctive hybrid MKNF knowledge bases is defined turning them into positive ones, thus allowing the application of the operator $T_{\mathcal{K}}$.

Definition 8 *Let $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ be a ground nondisjunctive DL-safe hybrid MKNF knowledge base and $S \subseteq \mathsf{KA}(\mathcal{K}_G)$. The* MKNF transform *$\mathcal{K}_G/S = (\mathcal{O}, \mathcal{P}_G/S)$ is obtained by $\mathcal{P}_G/S$ containing all rules $\mathbf{K}\, H \leftarrow \mathbf{K}\, A_1, \ldots, \mathbf{K}\, A_n$ for which there exists a rule $\mathbf{K}\, H \leftarrow \mathbf{K}\, A_1, \ldots, \mathbf{K}\, A_n, \mathbf{not}\, B_1, \ldots, \mathbf{not}\, B_m$ in $\mathcal{P}_G$ with $\mathbf{K}\, B_j \notin S$ for all $1 \leq j \leq m$.*

This resembles the transformation known from stable models of logic programs and the following operator using a fixpoint of $T_{\mathcal{K}}$ is thus straightforward to define.

Definition 9 *Let $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ be a nondisjunctive DL-safe hybrid MKNF knowledge base and $S \subseteq \mathsf{KA}(\widehat{\mathcal{K}})$. We define:*

$$\Gamma_{\mathcal{K}}(S) = T_{\mathcal{K}^*_G/S} \uparrow \omega$$

Even though we consider all modal atoms from $\mathsf{KA}(\widehat{\mathcal{K}})$, the applied knowledge base $\mathcal{K}^*_G$ does not enforce $\mathbf{not}\, H(t_1, \ldots, t_n)$ if $\neg H(t_1, \ldots, t_n)$ holds. Thus $\Gamma_{\mathcal{K}}$ alone is not sufficient to obtain the intended model and we define an operator similar in appearance but referring to $\widehat{\mathcal{K}}_G$.

Definition 10 *Let $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ be a nondisjunctive DL-safe hybrid MKNF knowledge base and $S \subseteq \mathsf{KA}(\widehat{\mathcal{K}})$. We define:*

$$\Gamma'_{\mathcal{K}}(S) = T_{\widehat{\mathcal{K}}_G/S} \uparrow \omega$$

Both, Γ and Γ', are shown to be antitonic and we join them as follows to two monotonic operators.

Definition 11 *Let $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ be a nondisjunctive DL-safe hybrid MKNF knowledge base and $S \subseteq \mathsf{KA}(\widehat{\mathcal{K}})$. We define:*

$$\Phi_{\mathcal{K}}(S) = \Gamma_{\mathcal{K}}(\Gamma'_{\mathcal{K}}(S)) \text{ and } \Psi_{\mathcal{K}}(S) = \Gamma'_{\mathcal{K}}(\Gamma_{\mathcal{K}}(S))$$

Since both are monotonic we obtain a least and a greatest fixpoint in both cases and the least fixpoint of $\Phi_{\mathcal{K}}$ and the greatest one of $\Psi_{\mathcal{K}}$ then define the well-founded partition.

Definition 12 *Let $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ be a nondisjunctive DL-safe hybrid MKNF knowledge base and let $\mathbf{P}_{\mathcal{K}}, \mathbf{N}_{\mathcal{K}} \subseteq \mathsf{KA}(\mathcal{K})$ with $\mathbf{P}_{\mathcal{K}}$ being the least fixpoint of $\Phi_{\mathcal{K}}$ and $\mathbf{N}_{\mathcal{K}}$ the greatest fixpoint of $\Psi_{\mathcal{K}}$, both restricted to the modal atoms only occurring in $\mathsf{KA}(\mathcal{K})$. Then $(P_W, N_W) = (\mathbf{P}_{\mathcal{K}} \cup \{\mathbf{K}\, \pi(\mathcal{O})\}, \mathsf{KA}(\mathcal{K}) \setminus \mathbf{N}_{\mathcal{K}})$ is the* well-founded partition *of $\mathcal{K}$.*

Both, $\mathbf{P}_{\mathcal{K}}$ and $\mathbf{N}_{\mathcal{K}}$, are restricted to the modal atoms occurring in $\mathcal{K}$ only. Thus, the auxiliary modal atoms introduced via $\widehat{\mathcal{K}}$ are not present in the well-founded partition. But they are not necessary there anyway since their only objective is preventing inconsistencies, i.e. deriving $\neg\varphi$ and $\mathbf{not}\, \varphi$ being undefined.

Example 2 *Consider the KB $\mathcal{K}$ from the example scenario and add the owners compilation $C2$, i.e. $owns(C2)$ and album $A3$ and compilation $C3$ which are both sufficiently different from $C2$. At first we add a DL statement $\neg O \sqsubseteq NO$ to the knowledge base to obtain $\mathcal{K}^*$. Additionally, each ground rule in $\widehat{\mathcal{K}}$ with head $O(x)$ receives a modal atom $\mathbf{not}\, NO(x)$ in its body where x here just functions as a placeholder for $A3$ and $C2$. When we compute Γ' of $\emptyset$ then the result contains $\mathbf{K}\, NO(C3)$, $\mathbf{K}\, O(C3)$ and of course $\mathbf{K}\, O(A3)$ and so does Γ applied to this result providing already the least fixpoint of $\Phi_{\mathcal{K}}$ (due to our simplifications). If we compute Γ of $\mathsf{KA}(\widehat{\mathcal{K}})$ then only $\mathbf{K}\, NO(C3)$ occurs due to $D_{\widehat{\mathcal{K}}}$ because all rules with modal $\mathbf{not}$-atoms are removed from the transform. However, computing Γ' of this result derives only additionally $\mathbf{K}\, O(A3)$ due to $\mathbf{not}\, NO(C3)$ occurring in the body of the rule with head $\mathbf{K}\, O(C3)$, and we also obtained the greatest fixpoint of $\Psi_{\mathcal{K}}$. So the well-founded partition contains $\mathbf{K}\, O(A3)$ in P_W, i.e. offers $A3$, but also $\mathbf{K}\, O(C3)$ in P_W and in N_W, i.e. $C3$ is offered and rejected at the same time. This is a clear indication that the knowledge base is inconsistent, as we shall see in the following section, and the (2) and (3) alone are not suitable to provide the intended service.*

5 Properties

In the same manner as done in [7] for the alternating fixpoint of normal logic programs, we restate the iteration for obtaining $\Phi_{\mathcal{K}}$ and $\Psi_{\mathcal{K}}$ as:

$$\begin{array}{lcllcl}
\mathbf{P}_0 & = & \emptyset & \mathbf{N}_0 & = & \mathsf{KA}(\widehat{\mathcal{K}}) \\
\mathbf{P}_{n+1} & = & \Gamma_{\mathcal{K}}(\mathbf{N}_n) & \mathbf{N}_{n+1} & = & \Gamma'_{\mathcal{K}}(\mathbf{P}_n) \\
\mathbf{P}_\omega & = & \bigcup \mathbf{P}_n & \mathbf{N}_\omega & = & \bigcap \mathbf{N}_n
\end{array}$$

It is easy to see that $\Phi_{\mathcal{K}} \uparrow 1 = \mathbf{P}_2$, $\Phi_{\mathcal{K}} \uparrow 2 = \mathbf{P}_4$, i.e. $\Phi_{\mathcal{K}} \uparrow i = \mathbf{P}_{2i}$, and likewise $\Psi_{\mathcal{K}} \downarrow i = \mathbf{P}_{2i}$. In particular, it can be shown that the sequence of $\mathbf{P}_i$ (respectively $\mathbf{N}_i$) is increasing, (respectively decreasing) and without surprise its limits concur with the least fixpoint of $\Phi_{\mathcal{K}}$, respectively the greatest fixpoint of $\Psi_{\mathcal{K}}$, i.e. $\mathbf{P}_\omega = \mathsf{lfp}(\Phi_{\mathcal{K}})$ and $\mathbf{N}_\omega = \mathsf{gfp}(\Psi_{\mathcal{K}})$[6]. As an overall benefit, we can compute the least fixpoint of $\Phi_{\mathcal{K}}$ directly from the greatest one of $\Psi_{\mathcal{K}}$ and vice versa.

Proposition 1 *Let $\mathcal{K}$ be a nondisjunctive DL-safe hybrid MKNF knowledge base. Then $\mathbf{P}_\omega = \Gamma(\mathbf{N}_\omega)$ and $\mathbf{N}_\omega = \Gamma'(\mathbf{P}_\omega)$.*

It furthermore can be shown that we can even use this computation to check consistency of the KB.

Proposition 2 *Let $\mathcal{K}$ be a nondisjunctive DL-safe hybrid MKNF knowledge base and $\mathbf{P}_\omega$ the least fixpoint of $\Phi_{\mathcal{K}}$. If $\Gamma'(\mathbf{P}_\omega) \subset \Gamma(\mathbf{P}_\omega)$ then $\mathcal{K}$ is MKNF-inconsistent.*

Intuitively, what this statement says is that any inconsistency between rules and the ontology can be discovered by computing the set of non-false modal atoms and then checking whether all false modal atoms are not just enforced to be false by the first-order knowledge although the (unchanged) rules support (at least one of) these false modal atoms. For an inconsistency check this is of course not sufficient, since an inconsistent DL base $\mathcal{O}$ is not detected by this method. In fact, in case we want to check for consistency of $\mathcal{K}$ we have both to check consistency of $\mathcal{O}$ alone, and apply the proposition above.

Theorem 1 *Let $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ be a nondisjunctive DL-safe hybrid MKNF knowledge base and $\mathbf{P}_\omega$ the least fixpoint of $\Phi_{\mathcal{K}}$. $\Gamma'(\mathbf{P}_\omega) \subset \Gamma(\mathbf{P}_\omega)$ or $\mathcal{O}$ is inconsistent iff $\mathcal{K}$ is MKNF-inconsistent.*

Normal rules alone cannot be inconsistent, unless we allow integrity constraints as rules whose head is $\mathbf{K}\, \mathbf{f}$ (cf. [11]). But then inconsistencies are easily detected since $\mathbf{P}_W$ or $\mathbf{N}_W$ contain $\mathbf{K}\, \mathbf{f}$.

Example 3 *Reconsider the result from Example 2. If we compute Γ of the least fixpoint of $\Phi_{\mathcal{K}}$, i.e. $\mathbf{P}_\omega$ then the result now contains $\mathbf{K}\, O(C3)$ while this is not contained in Γ' of $\mathbf{P}_\omega$. Our assumption that the KB is inconsistent is thus verified.*

However, in case of a consistent knowledge base, the well-founded partition always yields a three-valued model.

Theorem 2 *Let $\mathcal{K}$ be a consistent nondisjunctive DL-safe hybrid MKNF KB and $(\mathbf{P}_{\mathcal{K}} \cup \{\mathbf{K}\, \pi(\mathcal{O})\}, \mathsf{KA}(\mathcal{K}) \setminus \mathbf{N}_{\mathcal{K}})$ be the well-founded partition of $\mathcal{K}$. Then (I_P, I_N) where $I_P = \{I \mid I \models \mathsf{ob}_{\mathcal{K}, \mathbf{P}_{\mathcal{K}}}\}$ and $I_N = \{I \mid I \models \mathsf{ob}_{\mathbf{N}_{\mathcal{K}}}\}$ is an MKNF model – the* well-founded MKNF model.

In fact, the result is not any three-valued MKNF model but the least one with respect to the following order.

[6] $\Phi_{\mathcal{K}}$ and $\Psi_{\mathcal{K}}$ are order-continuous for the same reasons as $T_{\mathcal{K}}$.

Definition 13 *Let φ be a closed MKNF formula and (M_1, N_1) and (M_2, N_2) be partial MKNF models of φ. Then $(M_1, N_1) \succeq_k (M_2, N_2)$ iff $M_1 \subseteq M_2$ and $N_1 \supseteq N_2$.*

This order intuitively resembles the knowledge order where the least element contains the smallest amount of derivable knowledge, i.e. the one which leaves as much as possible undefined.

Theorem 3 *Let $\mathcal{K}$ be a consistent nondisjunctive DL-safe hybrid MKNF KB and (M, N) be the well-founded MKNF model. Then, for any three-valued MKNF model (M_1, N_1) of $\mathcal{K}$ we have $(M_1, N_1) \succeq_k (M, N)$.*

Moreover, for an empty DL base the well-founded partition corresponds to the well-founded model for (normal) logic programs.

Corollary 1 *Let $\mathcal{K}$ be a nondisjunctive program of MKNF rules, Π a normal logic program obtained from $\mathcal{P}$ by transforming each MKNF rule $\mathbf{K}\, H \leftarrow \mathbf{K}\, A_1, \ldots, \mathbf{K}\, A_n, \mathbf{not}\, B_1, \ldots, \mathbf{not}\, B_m$ into a clause $H \leftarrow A_1, \ldots, A_n, \mathbf{not}\, B_1, \ldots, \mathbf{not}\, B_m$ of Π, $W_{\mathcal{K}} = (P, N)$ be the well-founded MKNF model, and W_Π be the well-founded model of Π. Then $\mathbf{K}\, H \in P$ if and only if $H \in W_\Pi$ and $\mathbf{K}\, H \in N$ if and only if $\mathbf{not}\, H \in W_\Pi$.*

Finally the data complexity result is obtained basically from the result of $T_{\mathcal{K}}$ for positive nondisjunctive MKNF knowledge bases in [11] where data complexity is measured in terms of A-Box assertions and rule facts.

Theorem 4 *Let $\mathcal{K}$ be a nondisjunctive DL-safe hybrid MKNF KB. Assuming that entailment of ground DL-atoms in $\mathcal{DL}$ is decidable with data complexity $\mathcal{C}$ the data complexity of computing the well-founded partition is in $\mathrm{P}^{\mathcal{C}}$.*

This means that if the description logic fragment is tractable,[7] we end up with a formalism whose model is computed with a data complexity of P.

6 Comparisons and Conclusions

As already said, [11] is the stable model oriented origin of our work. The data complexity for reasoning with (2-valued) MKNF models in nondisjunctive programs is shown to be $\mathcal{E}^{\mathrm{P}^{\mathcal{C}}}$ where $\mathcal{E} = \mathrm{NP}$ if $\mathcal{C} \subseteq \mathrm{NP}$, and $\mathcal{E} = \mathcal{C}$ otherwise. Thus, computing the well-founded partition generally ends up in a strictly smaller complexity class than deriving one of maybe various MKNF models. However, M is a (two-valued) MKNF model of $\mathcal{K}$ iff (M, M) is a three-valued MKNF model of $\mathcal{K}$ and, furthermore, if (M, M) is the well-founded MKNF model of $\mathcal{K}$, M is the only MKNF model of $\mathcal{K}$. Furthermore, the well-founded partition can also be used in the algorithms presented in [11] for computing a subset of that knowledge which holds in all partitions corresponding to a two-valued MKNF model.

The approach presented in [8], though conceptually similar to ours, is based on a different semantics which evaluates $\mathbf{K}$ and $\neg\mathbf{not}$ (and $\neg\mathbf{K}$ and $\mathbf{not}$) simultaneously, thereby differing from the ideas of [11]. This in particular does not allow to minimize unnecessary undefinedness. Furthermore, in contrast with our approach, [8] does not allow for any form of detection of inconsistencies resulting from the interaction of the DL part and the rules. Instead, it provides a strange kind of model in these cases which contains undefined modal atoms which are actually first-order false. Therefore, our proposal is more robust than the one in [8] and in fact more closely related to the two-valued one.

In summary, here we define a WFS of (tightly integrated) hybrid KBs that is sound wrt. the semantics defined in [11] for MKNF KBs, that has strictly lower complexity, coinciding with it in case there are no rules, and that coincides with the WFS of normal programs [14] in case the DL-part is empty. We also obtain tractable fragments whenever the underlying DL is tractable. Moreover, we define a construction for computing the WF-model that is also capable of detecting inconsistencies.

It is worth noting that when inconsistencies come from the combination of the rules with the DL-part (i.e. for inconsistent KBs with a consistent DL-part), the construction still yields some results (e.g. in example 2). This suggests that the method could be further exploited in the direction of defining a paraconsistent semantics for hybrid KBs. This, together with a study of tractable fragments, generalization to disjunctive rules and implementations, are subjects for future work.

[7] See e.g. the W3C member submission on tractable fragments of OWL 1.1 at `http://www.w3.org/Submission/owl11-tractable/`.

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